Chapter 13

Satisficing DEA models under chance constraints

W.W. Cooper
Graduate School of Business, The University of Texas at Austin, Austin, TX 78712-1174, USA

Zhimin Huang and Susan X. Li
Schools of Business and Banking, Adelphi University, Garden City, Long Island, NY 11530, USA

DEA (Data Envelopment Analysis) models and concepts are formulated here in terms of the "P-Models" of Chance Constrained Programming, which are then modified to contact the "satisficing concepts" of H.A. Simon. Satisficing is thereby added as a third category to the efficiency/inefficiency dichotomies that have heretofore prevailed in DEA. Formulations include cases in which inputs and outputs are stochastic, as well as cases in which only the outputs are stochastic. Attention is also devoted to situations in which variations in inputs and outputs are related through a common random variable. Extensions include new developments in goal programming with deterministic equivalents for the corresponding satisficing models under chance constraints.

Keywords: Efficiency, satisficing, data envelopment analysis, stochastic efficiency.

1 Introduction

DEA (Data Envelopment Analysis), as initiated and developed by Charnes et al. (1978), is a nonparametric method for identifying efficient production frontiers and evaluating the relative efficiency of decision making units (DMUs), each of which is an entity responsible for converting multiple inputs into multiple outputs. Extensions with accompanying examples of uses of DEA can be found in references like Banker et al. (1984), Banker et al. (1989), Charnes and Cooper (1985) and Seiford and Thrall (1990). All of these DEA models are deterministic, but recent extensions have been directed to incorporating statistical or probabilistic considerations into one or more of the basic DEA models that have now been developed.

Banker (1986, 1993) incorporated statistical elements into DEA and developed a non-parametric approach with maximum likelihood methods used to effect inferences in the presence of statistical noise (see also Banker and Cooper (1994) for further extensions). Proceeding in a different direction, Sengupta (1982, 1987, 1988 and 1989) and Desai and Schinnar (1987) introduced chance constrained programming formulations of DEA, while Land et al. (1992, 1993, 1994) utilized chance constrained programming with accompanying developments that enabled them (1) to analyze new problems such as evaluating the relative efficiency of communist and capitalist systems as well as (2) to re-evaluate earlier deterministic applications of DEA such as the Charnes–Cooper-Rhodes (1981) study of the Program Follow Through experiment in U.S. public school education.

All of these chance constrained programming formulations utilize expected value optimizations and hence fall in the class that Charnes and Cooper (1963) refer to as "E-Models". Here we turn to the more general class that Charnes and Cooper refer to as "P-Models" and develop them in a manner that enables us to make contact with theories of behavior such as are described by the "satisficing concepts" of Simon (1957). In this way, we expand potential uses of DEA domains in social psychology while also maintaining contact with the capabilities for use in operations research and economics which have characterized preceding work in DEA.

The developments in this paper proceed as follows. In the next section, we introduce chance constrained programming models that enable us to define concepts of "stochastic efficiency", which we then interpret in terms of managerial "policies" and "rules". Next, we extend these efficiency formulations in a manner that enables us to include "satisficing" as well as "optimizing" (efficient) behavior which we distinguish from the inefficiencies that may be encountered in either case. These conceptual formulation are followed by mathematical developments with accompanying theorems and proofs that provide access to "deterministic equivalents" which can be used to obtain solutions to our chance constrained models when zero-order decision rules and normal distributions are appropriate. A concluding section discusses extensions and additional research that will be needed when other, more general, situations need to be dealt with.

2 Stochastic efficiency

We start by introducing the following version of a P-Model, which we use to adapt the usual definitions of "DEA efficiency" to a Chance Constrained Programming context,

$$
\begin{aligned}
&\text{maximize } P\left\{\frac{\sum_{r=1}^{s} u_r \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i \tilde{x}_{io}} \geq 1\right\} \\
&\text{subject to } P\left\{\frac{\sum_{r=1}^{s} u_r \tilde{y}_{rj}}{\sum_{i=1}^{m} v_i \tilde{x}_{ij}} \geq 1\right\} \geq 1-\alpha_j, \quad j=1,\ldots,n, \\
&\qquad u_r, v_i \geq 0 \quad \forall r, i.
\end{aligned} \tag{1}
$$

Here, P means "Probability" and the symbol "~" is used to identify the inputs and outputs as random variables with a known joint probability distribution. The u_r,

Co
De
Jag
Kal
Lar
Lan
Lan
Leit
Rho
Seif
Seng
Seng
Seng
Seng
Sharp
Simo
Stedr
Stigle
Wagn

$v_i \geq 0$ are the virtual multipliers (= weights) to be determined by solving the above problem. This model evidently builds upon the CCR model of DEA, with the ratio in the objective representing output and input values for DMU_o, the DMU to be evaluated, which is also included in the $j = 1,\ldots,n$ DMUs with output-to-input ratios represented as chance constraints.

Evidently, the constraints in (1) are satisfied by choosing $u_r = 0$, and $v_i > 0$ for all r and i. Hence, for continuous distributions like the ones considered in this paper, it is not vacuous to write

$$\Pr\left\{\frac{\sum_{r=1}^{s} u_r^* \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i^* \tilde{x}_{io}} \leq 1\right\} + \Pr\left\{\frac{\sum_{r=1}^{s} u_r^* \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i^* \tilde{x}_{io}} \geq 1\right\} = 1$$

or (2)

$$\Pr\left\{\frac{\sum_{r=1}^{s} u_r^* \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i^* \tilde{x}_{io}} \leq 1\right\} = 1 - \alpha^* \geq 1 - \alpha_o,$$

$$u_r, v_i \geq 0 \quad \forall r, i.$$

Here, * refers to an optimal value, so α^* is the probability of achieving a value of at least unity with this choice of weights and $1 - \alpha^*$ is therefore the probability of failing to achieve this value.

To see how these formulations may be used, we note that we must have $\alpha_o \geq \alpha^*$, since $1 - \alpha_o$ is prescribed in the constraint for $j = o$ as the chance allowed for characterizing the $\tilde{y}_{ro}, \tilde{x}_{io}$ values as inefficient. More formally, we introduce the following stochasticized definition of efficiency:

Definition
DMU_o is "stochastic efficient" if and only if $\alpha^* = \alpha_o$.

This opens a variety of new directions for research and potential uses of DEA. Before indicating some of these possibilities, however, we replace (1) with the following:

$$\begin{aligned}
&\text{maximize } \Pr\left\{\frac{\sum_{r=1}^{s} u_r \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i \tilde{x}_{io}} \geq 1\right\} \\
&\text{subject to } \Pr\left\{\frac{\sum_{r=1}^{s} u_r \tilde{y}_{rj}}{\sum_{i=1}^{m} v_i \tilde{x}_{ij}} \leq 1\right\} + \Pr\left\{\frac{\sum_{r=1}^{s} u_r \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i \tilde{x}_{io}} \geq 1\right\} \geq 1, \quad j = 1,\ldots,n, \\
&\qquad u_r, v_i \geq 0 \quad \forall r, i.
\end{aligned} \tag{3}$$

This simpler model makes it easier to see what is involved in uses of these CCP/DEA formulations. It also enables us to examine potential uses in a simplified manner.

First, as is customary in CCP, it is assumed that the behavior of the random variables are governed by a known multivariate distribution. Hence we can examine the value of α^* even before the data are generated. If this value is too small, then one can signal central management, say, that the situation for DMU_o needs to be examined in advance because there is a probability of at least $1 - \alpha^* \geq 1 - \alpha_o$ that it will not perform efficiently.

Some additional uses of these concepts can be brought into view from the original work in CCP. For instance, the article by Charnes et al. (1958) which introduced CCP was concerned with policies and programs involved in scheduling heating oil production for EXXON (then known as Standard Oil of New Jersey). This led to the formation of a risk evaluation committee (the first in the Company's history) to determine suitable choices of α^*. It was decided that a "policy" to supply all customers on demand would require $\alpha^* > 1/2$ since the alternate choice of $\alpha^* \leq 1/2$ was likely to be interpreted by customers, and others, to mean that the company was either indifferent or unlikely to be willing to supply all customers on demand.[1)]

If we define a "rule" as "a chance constraint which is to hold with probability one", then we can regard a "policy" as "a chance constraint which is to hold with probability $0.5 < \alpha^* < 1$". Implementation of a "policy" allows for deviations which can require managerial attention whereas a "rule" may be administered in clerical fashion since no exceptions are to be permitted. Notice, too, that a policy may be identified and evaluated by reference to ex-post data, as in an accounting or performance audit, in order to see whether the corresponding actions had been taken sufficiently frequently, or whether some "policy" other than the intended one had prevailed. See the definitions and discussions of the term "audit" in Cooper and Ijiri (1983).

We can now bring the above discussion into focus for its possible use in efficiency evaluations because the constraint for $j = o$ in (3) contains complementary possibilities. Hence, accepting a value of $\alpha^* > 1/2$ means acceptance of a policy that favors efficient performance, whereas a value of $\alpha^* \leq 1/2$ means that indifferent or inefficient performance is favored. This does not end the matter. The already calculated u_r^*, v_i^* remain available for use and may also be applied to the data that materialize after operations are undertaken by DMU_o. Applying the previously determined weights to the thus generated data allows us to calculate the probability that the values realized by DMU_o will occur. Using these weights, we may then determine whether the observed inputs and outputs yield a ratio that is within the allowable range of probabilities or whether a shift in the initially assumed multivariate distribution has occurred.

1) This characterization and usage of the term "policy" was important because the company was especially concerned with heating oil as a "commodity charged with a public interest" since failure to supply it to customers on demand (in cold weather) could have severe consequences. See the discussion of this "policy" in Charnes et al. (1958).

Further pursuit of this topic would lead into discussions of the higher-order decision rules in CCP and/or the use of Bayesian procedures to modify the initially assumed probability distributions. See, e.g., Jaganathan (1985). We do not follow this route but prefer, instead, to move toward extensions of (1) that will enable us to make contact with the "satisficing" concepts of Simon (1957) that were promised in our opening section.

3 Optimizing and satisficing

The following model represents an evident generalization of (1):

$$
\begin{aligned}
&\text{maximize} \ \Pr\left\{ \frac{\sum_{r=1}^{s} u_r \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i \tilde{x}_{io}} \geq \beta_o \right\} \\
&\text{subject to} \ \Pr\left\{ \frac{\sum_{r=1}^{s} u_r \tilde{y}_{rj}}{\sum_{i=1}^{m} v_i \tilde{x}_{ij}} \leq \beta_j \right\} \geq 1 - \alpha_j \quad j = 1, \ldots, n, \\
&\Pr\left\{ \frac{\sum_{r=1}^{s} u_r \tilde{y}_{rj}}{\sum_{i=1}^{m} v_i \tilde{x}_{ij}} \geq \beta_j \right\} \geq 1 - \alpha_j \quad j = n+1, \ldots, n+k, \\
&u_r, v_i \geq 0 \quad \forall r, i.
\end{aligned}
\tag{4}
$$

Here we interpret β_o as an "aspiration level" either imposed by an outside authority, as in the budgeting model of Stedry (1960), or adopted by an individual for some activity as in the satisficing concept of Simon (1957). We may then think of the first $j = 1, \ldots, n$ constraints as representing various conditions such as physical possibilities or the endurance limits of this individual. The added constraints $j = n+1, \ldots, n+k$ may represent further refinements of the aspiration levels. This could even include a constraint $\beta_j = \beta_o$ with a prescribed level of probability for achieving this aspired level that might exceed the maximum possible value. The problem would then have no solution, and this would activate psychological mechanisms, as described by Simon (1957), in which an individual must revise his or her aspirations and/or willingness to accept risks of not achieving them.

Uses of these ideas in actual applications are yet to be made. However, we think that potential uses include possibilities for using DEA to extend the kinds of behavior that are represented in the approaches used in both economics and psychology. For instance, it is now common to contrast "satisficing" and "optimizing behavior" as though the two are mutually exclusive. Reformulation and use of DEA along lines like we have just described, however, may enable us to discover situations in which both types of behavior might be present. Indeed, it is possible that behaviors which are now characterized as inefficient (with deterministic formulations) might better be interpreted as examples of satisficing behavior with associated probabilities of occurrence. This kind of characterization may, in turn, lead to further distinctions in which satisficing gives way to inefficiencies when probabilities are too low even for satisficing behavior and this, we think, provides access to sharper and better possibilities than those offered in the economics literature by Stigler's (1976) critique of Leibenstein's (1976) concept of "X-Efficiency".

The preceding interpretations were pointed toward individual behaviors that accord with the satisficing characterizations provided in Simon (1957). Turning now to managerial uses, we can simplify matters by eliminating the last k constraints in (4) from consideration. Then we can interpret the above problem in a manner that differs from (1) because we have estimates (or are otherwise willing to assume) that values of $\beta_j < 1$ are applicable for each of the $j = 1,\ldots,n$ DMUs to be considered. One of the DMUs of interest is then singled out for evaluating the probability that its performance will exceed the $\beta_j = \beta_o$ value assigned to (or assumed for) this entity in the constraints. Proceeding as we did in our discussion of (1), we can then interpret our results as being applicable in either an ex ante or ex post manner according to whether our interest is in planning or control. In a planning mode, for instance, we can determine a maximum probability of inefficient or satisficing (tolerably inefficient) behavior that we may want to anticipate or forestall when $\alpha^* < \alpha_o$ occurs. For control purpose, we may similarly want to determine whether the observed behavior, as recorded, is too far out for us to regard it as having conformed to what should have occurred.

4 Deterministic equivalents

The above models are very general and intended mainly for conceptual interpretation. They can also provide guidance for the more specialized developments that we now undertake to achieve "deterministic equivalents" for computation and implementation in applicable circumstances.

Following Land et al. (1992, 1993, 1994), we assume that input values are deterministic so that only the outputs are to be represented as random variables with a multivariate normal distribution and known parameters. Again like Land et al., we restrict attention to the class of zero-order decision rules.

The class of zero-order decision rules for use in chance constrained programming can be most easily explained by returning to the example of scheduling heating oil production at EXXON, where the objective was to secure a best schedule for this seasonal (weather-dependent) product as required to anticipate the probabilistic demands. Decisions rules were developed that allowed for changing schedules, in conditional stochastic fashion, as sales materialized. A zero-order rule, however, would have set the schedules for the entire season and use of this rule means that the vectors u and v of multipliers in (1) are to be treated as deterministic variables.

Choices of multivariate normal distributions and zero-order decision rules are less restrictive than might at first appear to be the case. Transformations are available for bringing other types of distributions into approximately normal form – as is done in Charnes et al. (1968), for instance, when it was found necessary to treat the case of highly skewed (log-normal) distributions which were encountered when developing new product marketing strategies. We can also adapt our use of zero-order decision rules by interpreting them as a series of one-period-at-a-time applications with appropriate models, to allow for changing realizations and probabilities, and regard these (in many situations) as approximations to the more complex solution procedures involved in developing higher order "conditional" decision rules to deal with full-scale treatment of the dynamics. Proceeding in this one-period-at-a-time manner also allows us to bypass additional problems such as the sample size considerations which are encountered in dealing with multiple observations. See Charnes et al. (1986) for treatments of sample sizes in CCP.

We start by replacing $\tilde{x}_j$ as defined in (1) with x_j which means that the inputs are predetermined, and hence are no longer random, and we also suppose that all components of each x_j vector are positive. Then, using $\bar{y}_j, j = 1,\ldots,n$, to represent the vector of means for $\tilde{y}_j$, we can obtain

$$P\left\{\frac{u^T\tilde{y}_j}{v^T x_j} \leq \beta_j\right\} = P\{u^T\tilde{y}_j \leq \beta_j v^T x_j\}$$

$$= P\left\{\frac{u^T\tilde{y}_j - u^T\bar{y}_j}{\sqrt{u^T\Sigma_j u}} \leq -\frac{u^T\bar{y}_j - \beta_j v^T x_j}{\sqrt{u^T\Sigma_j u}}\right\}, \tag{5}$$

where $\Sigma_j = (\mathrm{Cov}(\tilde{y}_{ij}, \tilde{y}_{kj}))$ is positive, with the symbol "Cov" referring to the covariance operator.

Next we introduce new variables $\tilde{z}_j$, defined by

$$\tilde{z}_j = \frac{u^T\tilde{y}_j - u^T\bar{y}_j}{\sqrt{u^T\Sigma_j u}}, \quad j = 1,\ldots,n, \tag{6}$$

so that $\tilde{z}_j$ follows the standard normal probability distribution (with zero mean and unit variance). Direct substitution in (5) then gives

$$P\left\{\tilde{z}_j \leq -\frac{u^T\bar{y}_j - \beta_j v^T x_j}{\sqrt{u^T\Sigma_j u}}\right\} \geq 1 - \alpha_j \tag{7}$$

for each of the first $j = 1,\ldots,n$ constraints we are treating in (4).

Since $\tilde{z}_j$ follows the standard normal probability distribution we use its property of invertibility to rewrite (7) as

$$-\frac{u^T \bar{y}_j - \beta_j v^T x_j}{\sqrt{u^T \Sigma_j u}} \geq \Phi^{-1}(1-\alpha_j), \tag{8}$$

where Φ is the standard normal distribution function and Φ^{-1}, its inverse, is the so-called "fractile function".

Proceeding as in Charnes and Cooper (1963), we introduce nonnegative "spacer variables", η_j, and replace (8) with the following two inequalities:

$$\begin{aligned} u^T \bar{y}_j - \beta_j v^T x_j - \Phi^{-1}(\alpha_j)\eta_j &\leq 0, \\ C_{\alpha_j}[\eta_j^2 - u^T \Sigma_j u] &\geq 0, \end{aligned} \tag{9}$$

where

$$C_{\alpha_j} = \begin{cases} 1 & \text{if } \alpha_j < 0.5, \\ 0 & \text{if } \alpha_j = 0.5, \\ -1 & \text{if } \alpha_j > 0.5. \end{cases}$$

Considering only the first n constraints in (4) and replacing $\tilde{x}_j$ with x_j, we then have

$$\begin{aligned} &\text{Maximize} \quad P\left\{\frac{u^T \tilde{y}_0}{v^T x_0} \geq \beta_0\right\} \\ &\text{subject to} \quad u^T \bar{y}_j - \beta_j v^T x_j - \Phi^{-1}(\alpha_j)\eta_j \leq 0, \quad j = 1, \dots, n, \\ &\qquad C_{\alpha_j}[\eta_j^2 - u^T \Sigma_j u] \geq 0, \quad j = 1, \dots, n, \\ &\qquad u \geq 0, \ v \geq 0, \ \eta \geq 0. \end{aligned} \tag{10}$$

The constraints in (10) are all deterministic but the functional still involves the vector $\tilde{y}_0$ of random variables, so this problem is not yet deterministic. It is easy to see, however, that (10) is equivalent to

$$\begin{aligned} &\text{Maximize} \quad \gamma \\ &\text{subject to} \quad P\left\{\frac{u^T \tilde{y}_0}{v^T x_0} \geq \beta_0\right\} \geq \gamma, \\ &\qquad u^T \bar{y}_j - \beta_j v^T x_j - \Phi^{-1}(\alpha_j)\eta_j \leq 0, \quad j = 1, \dots, n, \\ &\qquad C_{\alpha_j}\left\lfloor \eta_j^2 - u^T \Sigma_j u \right\rfloor \geq 0, \quad j = 1, \dots, n, \\ &\qquad u \geq 0, \ v \geq 0, \ \eta \geq 0. \end{aligned} \tag{11}$$

We can achieve a deterministic equivalent for the first constraint in (11) by proceeding as follows:

$$P\left\{\frac{u^T\tilde{y}_0}{v^T x_0} \geq \beta_0\right\} = P\{-u^T\tilde{y}_0 \leq -\beta_0 v^T x_0\}$$

$$= P\left\{\frac{-u^T\tilde{y}_0 + u^T\bar{y}_0}{\sqrt{u^T\Sigma_0 u}} \leq -\frac{\beta_0 v^T x_0 - u^T\bar{y}_0}{\sqrt{u^T\Sigma_0 u}}\right\}$$

$$= P\left\{\tilde{z}_0 \leq \frac{u^T\bar{y}_0 - \beta_0 v^T x_0}{\sqrt{u^T\Sigma_0 u}}\right\}. \tag{12}$$

For the first constraint in (11), we therefore have

$$P\left\{\tilde{z}_0 \leq \frac{u^T\bar{y}_0 - \beta_0 v^T x_0}{\sqrt{u^T\Sigma_0 u}}\right\} \geq \gamma. \tag{13}$$

This is equivalent to

$$\frac{u^T\bar{y}_0 - \beta_0 v^T x_0}{\sqrt{u^T\Sigma_0 u}} \geq \Phi^{-1}(\gamma). \tag{14}$$

Utilizing these results, we then find that (11) is equivalent to

$$\begin{aligned}
&\text{Maximize} \quad \gamma\\
&\text{subject to} \quad \frac{u^T\bar{y}_0 - \beta_0 v^T x_0}{\sqrt{u^T\Sigma_0 u}} \geq \Phi^{-1}(\gamma),\\
&\quad u^T\bar{y}_j - \beta_j v^T x_j - \Phi^{-1}(\alpha_j)\eta_j \leq 0, \quad j = 1,\dots,n,\\
&\quad C_{\alpha_j}\left\lfloor \eta_j^2 - u^T\Sigma_j u \right\rfloor \geq 0, \quad j = 1,\dots,n,\\
&\quad u \geq 0,\ v \geq 0,\ \eta \geq 0.
\end{aligned} \tag{15}$$

This problem is deterministic but is awkward to work with because the denominator in the first constraint makes this a non-convex programming problem. Our next objective is to remove this difficulty and we begin by considering the following problem,

$$\begin{aligned}
&\text{Maximize} \quad \xi\\
&\text{subject to} \quad \frac{u^T\bar{y}_0 - \beta_0 v^T x_0}{\sqrt{u^T\Sigma_0 u}} \geq \xi,\\
&\quad u^T\bar{y}_j - \beta_j v^T x_j - \Phi^{-1}(\alpha_j)\eta_j \leq 0, \quad j = 1,\dots,n,\\
&\quad C_{\alpha_j}\left\lfloor \eta_j^2 - u^T\Sigma_j u \right\rfloor \geq 0, \quad j = 1,\dots,n,\\
&\quad u \geq 0,\ v \geq 0,\ \eta \geq 0.
\end{aligned} \tag{16}$$

Since $\Phi^{-1}(\gamma)$, as represented in (14), is a strictly increasing function of γ, (15) and (16) have the same solution structure and, at any pair of optimal solutions for these two problems, we have

$$\xi^* = \Phi^{-1}(\gamma^*), \tag{17}$$

where * refers to an optimal value. It is now easy to see that (16) is equivalent to the following problem:

$$\begin{aligned}
&\text{Maximize} && \frac{u^T\bar{y}_0 - \beta_0 v^T x_0}{\sqrt{u^T\Sigma_0 u}} && \\
&\text{subject to} && u^T\bar{y}_j - \beta_j v^T x_j - \Phi^{-1}(\alpha_j)\eta_j \le 0, && j = 1,\dots,n, \\
& && C_{\alpha_j}\left\lfloor \eta_j^2 - u^T\Sigma_j u \right\rfloor \ge 0, && j = 1,\dots,n, \\
& && u \ge 0,\ v \ge 0,\ \eta \ge 0. &&
\end{aligned} \tag{18}$$

Since $(u^T\bar{y}_0 - \beta_0 v^T x_0)/\sqrt{u^T\Sigma_0 u}$ is bounded by $\Phi^{-1}(\alpha_0)$, it is easy to show that, by introducing a positively valued variable, ω, (18) is equivalent to

$$\begin{aligned}
&\text{Maximize} && \frac{u^T\bar{y}_0 - \beta_0 v^T x_0}{\omega} && \\
&\text{subject to} && u^T\Sigma_0 u - \omega^2 \ge 0, && \\
& && u^T\bar{y}_j - \beta_j v^T x_j - \Phi^{-1}(\alpha_j)\eta_j \le 0, && j = 1,\dots,n, \\
& && C_{\alpha_j}\left\lfloor \eta_j^2 - u^T\Sigma_j u \right\rfloor \ge 0, && j = 1,\dots,n, \\
& && u \ge 0,\ v \ge 0,\ \eta \ge 0,\ \omega \ge 0. &&
\end{aligned} \tag{19}$$

This problem involves a fractional functional in the objective. Hence, we can utilize the Charnes–Cooper transformation of linear fractional programming (see Charnes and Cooper (1962)) for which we let $t = 1/\omega$, $\mu := tu$, $\nu := tv$, and $\zeta := t\eta$. We can then replace (19) by the following quadratic programming problem:

$$\begin{aligned}
&\text{Maximize} && \mu^T\bar{y}_0 - \beta_0 \nu^T x_0 && \\
&\text{subject to} && u^T\Sigma_0 \mu \ge 1, && \\
& && \mu^T\bar{y}_j - \beta_j \nu^T x_j - \Phi^{-1}(\alpha_j)\zeta_j \le 0, && j = 1,\dots,n, \\
& && C_{\alpha_j}\left\lfloor \zeta_j^2 - u^T\Sigma_j u \right\rfloor \ge 0, && j = 1,\dots,n, \\
& && \mu \ge 0,\ \nu \ge 0,\ \zeta \ge 0. &&
\end{aligned} \tag{20}$$

We now conclude these developments by relating the solutions of (20) to the portion of (4) which we are dealing with via the following

Theorem 1
Let (μ^*, ν^*, ζ^*) and (u^*, v^*) be optimal solutions of (20) and (4), respectively; then

$$\Phi(\mu^{*T}\bar{y}_0 - \beta_0 \nu^{*T} x_0) = P\left\{ \frac{u^{*T}\tilde{y}_0}{v^{*T}x_0} \geq \beta_0 \right\}.$$

Furthermore, DMU$_o$ is stochastically efficient if and only if $\Phi(\mu^{*T}\bar{y}_0 - \beta_0 \nu^{*T} x_0) = \alpha_0$.

Proof
From (17), we have $\Phi(\mu^{*T}\bar{y}_0 - \beta_0 \nu^{*T} x_0) = \gamma^*$, where γ^* is the optimal value of problem (15). Since problem (15) is equivalent to problem (4),

$$\gamma^* = P\left\{ \frac{u^{*T}\tilde{y}_0}{v^{*T}x_0} \geq \beta_0 \right\}$$

and

$$\Phi(\mu^{*T}\bar{y}_0 - \beta_0 \nu^{*T} x_0) = P\left\{ \frac{u^{*T}\tilde{y}_0}{v^{*T}x_0} \geq \beta_0 \right\}.$$

□

5 Dualities under a single stochastic index factor

We now further specialize our analysis and assume that components of the outputs are related only through some basic underlying factor.[2)] All components of any output are determined solely by this single factor. More explicitly,

$$\tilde{y}_{rj} = \bar{y}_{rj} + b_{rj}\xi \tag{21}$$

for $r = 1,\ldots,s$ and $j = 1,\ldots,n$, where ξ, the level of some index (such as GNP), follows a normal distribution with $E(\xi) = 0$ and standard deviation $\sigma(\xi)$, while $\bar{y}_{rj}$ and b_{rj}, which are positive parameters, represent the expected values and standard deviations, respectively, for $\tilde{y}_{rj}$.

Without loss of generality, we now assume that $\sigma(\xi) = 1$ and consider how to achieve a deterministic equivalent for (4) under this single random index assumption. Employing algebraic reductions and analyses that are analogous to those given in section 4, we achieve

2) The use of a single underlying factor assumption has been recognized and applied for a long time in finance and economics. See, for example, Sharpe (1963) and Kahane (1977).

$$\begin{aligned} \text{Maximize} \quad & \mu^T \bar{y}_0 - \beta_0 v^T x_0 \\ \text{subject to} \quad & \mu^T b_0 = 1, \\ & \beta_j v^T x_j - \mu^T \bar{y}_j \geq \Phi^{-1}(1-\alpha_j)\mu^T b_j, \quad j = 1, \ldots, n, \\ & \mu \geq 0, \ v \geq 0. \end{aligned} \tag{22}$$

The dual of (22) is as follows:

$$\begin{aligned} \text{Minimize} \quad & -\theta \\ \text{subject to} \quad & \sum_{j=1}^{n} \lambda_j [\bar{y}_j + \Phi^{-1}(1-\alpha_j) b_j] \geq \bar{y}_0 + \theta b_0, \\ & \sum_{j=1}^{n} \lambda_j \beta_j x_j \leq \beta_0 x_0, \\ & \lambda \geq 0. \end{aligned} \tag{23}$$

By duality theory, for any optimal solution (λ^*, θ^*) of (23), we have $-\theta^* = \mu^{*T} y_0 - \beta_0 v^{*T} x_0$ and therefore $\Phi(-\theta^*) = \Phi(\mu^{*T}\bar{y}_0 - \beta_0 v^{*T} x_0) \leq \alpha_0$, with this last result following from the constraint for $j = o$ in (22). Hence, we have

$$\theta^* \geq \Phi^{-1}(1-\alpha_0). \tag{24}$$

With these results in hand, we are now able to draw some conclusions about the solutions. First we observe that the first and the last constraints in (23) together imply that only nonnegative combinations of deviations above or below the means of $\tilde{y}_j$ by amounts equal to $\Phi^{-1}(1-\alpha_j)b_j$ are considered, where $\Phi^{-1}(1-\alpha_j)$ are fractiles corresponding to α_j.[3] This means that output production must equal or exceed the mean of $\tilde{y}_0$ by an amount equal to θb_0. The second and the third constraints in (23) together imply that only nonnegative combinations of observed input levels are considered, and such a weighted sum must reflect input consumption not exceeding the observed level for DMU$_o$ all multiplied by their β_j, β_0 values.

The standard normal deviate $\bar{y}_0 + \theta b_0$ above or below[4] mean $\bar{y}_0$ can be used in the following manner to obtain an efficiency score for DMU$_o$: "DMU$_o$ is inefficient if $\theta^* > \Phi^{-1}(1-\alpha_0)$, while DMU$_o$ is efficient if $\theta^* = \Phi^{-1}(1-\alpha_0)$".

[3] Whether it is below or above depends on the value of α_j. If $\alpha_j > 0.5$, then $\bar{y}_j + \Phi^{-1}(1-\alpha_j)b_j$ is $-\Phi^{-1}(1-\alpha_j)$ times the standard deviation below $\bar{y}_j$. Otherwise, $\bar{y}_j + \Phi^{-1}(1-\alpha_j)b_j$ is $-\Phi^{-1}(1-\alpha_j)$ times the standard deviation above $\bar{y}_j$.

[4] If θ is positive, $\bar{y}_0 + \theta b_0$ is θ times the value of b_0 above mean y_0. Otherwise, $\bar{y}_0 + \theta b_0$ is $-\theta$ times b_0 below y_0.

6 Deterministic equivalents under the assumption that outputs and inputs are both stochastic

We now reverse direction and point our analyses to the case where outputs as well as inputs follow multivariate normal distributions with finite means and covariances. This means that we allow variations to occur in the production process and we also allow data measurement and specification errors to occur so the efficient frontier can vary stochastically across DMUs.

We first assume that the output-input data set follows a multivariate normal distribution. Let $\bar{y}_j$ and $\bar{x}_j$ be expected values of $\tilde{y}_j$ and $\tilde{x}_j$, respectively, with $\bar{y}_j > 0$ and $\bar{x}_j > 0$. Using analyses similar to those employed in section 4, we obtain

$$\begin{aligned} \text{Maximize} \quad & \mu^T \bar{y}_0 - \beta_0 v^T \bar{x}_0 \\ \text{subject to} \quad & \mu^T \Sigma_0^{\text{OO}} \mu - 2\beta_0 \mu^T \Sigma_0^{\text{OI}} v + \beta_0^2 v^T \Sigma_0^{\text{II}} v \geq 1, \\ & \mu^T \bar{y}_j - \beta_j v^T \bar{x}_j - \Phi^{-1}(\alpha_j)\zeta_j \leq 0, \qquad j = 1,\ldots,n, \\ & C_{\alpha_j}[\mu^T \Sigma_j^{\text{OO}} \mu - 2\beta_j \mu^T \Sigma_j^{\text{OI}} v + \beta_j^2 v^T \Sigma_j^{\text{II}} v - \zeta_j^2] \leq 0, \quad j = 1,\ldots,n, \\ & \mu \geq 0, \ v \geq 0, \ \zeta \geq 0, \end{aligned} \tag{25}$$

where, for $j = 1,\ldots,n$,

$$\Sigma_j^{\text{OO}} = (\text{Cov}(\tilde{y}_{ij}, \tilde{y}_{kj}))_{s \times s}, \tag{26}$$

$$\Sigma_j^{\text{OI}} = (\text{Cov}(\tilde{y}_{ij}, \tilde{x}_{kj}))_{s \times m} = (\Sigma_j^{\text{IO}})^T, \tag{27}$$

$$\Sigma_j^{\text{II}} = (\text{Cov}(\tilde{x}_{ij}, \tilde{x}_{kj}))_{m \times m}. \tag{28}$$

We now extend our previous theorems and definitions of stochastic efficiency as follows.

Theorem 2
Let (μ^*, v^*, ζ^*) and (u^*, v^*) be optimal solutions of (25) and (4), respectively; then

$$\Phi(\mu^{*T}\bar{y}_0 - \beta_0 v^{*T}\bar{x}_0) = P\left\{ \frac{u^{*T}\tilde{y}_0}{v^{*T}\tilde{x}_0} \geq \beta_0 \right\}.$$

Furthermore, DMU$_o$ is stochastically efficient if and only if $\Phi(\mu^{*T}\bar{y}_0 - \beta_0 v^{*T}\bar{x}_0) = \alpha_0$.

Now we proceed in a manner analogous to the preceding section and assume that the components of the inputs and outputs are related only through a basic underlying factor. More explicitly, the component of any input and output is determined solely by a single factor:

$$\tilde{x}_{ij} = \bar{x}_{ij} + a_{ij}\xi, \tag{29}$$

$$\tilde{y}_{rj} = \bar{y}_{rj} + b_{rj}\xi, \tag{30}$$

for $i = 1,\dots,m$, $j = 1,\dots,n$ and $r = 1,\dots,s$, where ξ follows a normal distribution with mean $E(\xi)$ and standard deviation $\sigma(\xi) = 1$ and $\bar{x}_{ij}, \bar{y}_{rj}, a_{ij}$ and b_{rj} are positive constants. It is easy to see that $\bar{x}_{ij}$ and $\bar{y}_{rj}$ are the expected values for $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$, respectively, and a_{ij} and b_{rj} are standard deviations for $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$, respectively.

Employing analyses similar to those used in section 4, we have

$$\begin{aligned} &\text{Maximize} && \mu^T\bar{y}_0 - \beta_0 v^T\bar{x}_0 \\ &\text{subject to} && |\beta_0 v^T a_0 - \mu^T b_0| \ge 1, \\ & && \beta_j v^T\bar{x}_j - \mu^T\bar{y}_j \ge \Phi^{-1}(1-\alpha_j)|\beta_j v^T a_j - \mu^T b_j|, \quad j = 1,\dots,n, \\ & && \mu \ge 0,\ v \ge 0. \end{aligned} \tag{31}$$

There are absolute values in the constraints, so problem (31) is not an ordinary linear programming problem. However, we can use the goal programming theory developed by Charnes and Cooper (1961, 1977) to transform problem (31) into an ordinary quadratic programming problem.

Consider the expression $|\beta_j v^T a_j - \mu^T b_j|$. If $\beta_j v^T a_j - \mu^T b_j \ge 0$, let $\eta_j^+ = \beta_j v^T a_j - \mu^T b_j$, otherwise $\eta_j^- = -(\beta_j v^T a_j - \mu^T b_j)$. Hence, $|\beta_j v^T a_j - \mu^T b_j|$ can be expressed by $\eta_j^+ + \eta_j^-$, where η_j^+ and η_j^- satisfy $\beta_j v^T a_j - \mu^T b_j = \eta_j^+ - \eta_j^-$, $\eta_j^+\eta_j^- = 0$ and $\eta_j^+ \ge 0$, $\eta_j^- \ge 0$, as required for goal programming. The inequalities must be satisfied for any solution. Then we can use

$$\begin{aligned} \eta_0^+ + \eta_0^- &\ge 1, \\ \beta_0 v^T a_0 - \mu^T b_0 &= \eta_0^+ - \eta_0^-, \\ \eta_0^+\eta_0^- &= 0, \\ \eta_0^+ \ge 0,\ \eta_0^- &\ge 0 \end{aligned}$$

to replace the first constraint in (31), and use

$$\begin{aligned} \beta_0 v^T a_0 - \mu^T b_0 &= \eta_0^+ - \eta_0^-, \\ \beta_j v^T a_j - \mu^T b_j &= \eta_j^+ - \eta_j^-, \\ \eta_j^+\eta_j^- &= 0, \\ \eta_j^+ \ge 0,\ \eta_j^- &\ge 0 \end{aligned}$$

to replace the second constraint in (31) for each j.

Hence, problem (31) is equivalent to the following quadratic programming problem:

$$
\begin{aligned}
\text{Maximize} \quad & \mu^T \bar{y}_0 - \beta_0 v^T \bar{x}_0 \\
\text{subject to} \quad & \eta_0^+ + \eta_0^- \geq 1, \\
& \beta_j v^T \bar{x}_j - \mu^T \bar{y}_j \geq \Phi^{-1}(1-\alpha_j)(\eta_j^+ + \eta_j^-), \quad j = 1,\dots,n, \\
& \beta_j v^T a_j - \mu^T b_j = \eta_j^+ - \eta_j^-, \quad j = 1,\dots,n, \\
& \eta_j^+ \eta_j^- = 0, \quad j = 1,\dots,n, \\
& \mu, v, \eta^+, \eta^- \geq 0.
\end{aligned}
\tag{32}
$$

Problem (32) is referred to as in efficiency analysis form with the maximization directed to the choices of μ and v which yield the largest value of a satisfactory probability of achieving an aspiration ratio level of weighted outputs to weighted inputs allowed by the constraints. In this way, we retain contact with the earlier discussion of both "satificing" concepts and DEA frontiers.

7 Concluding remarks

We previously described a variety of uses for these developments, which range from classification and control of policies and performances and extend to studies of behavior ranging from "satisficing" to "optimizing" possibilities in various situations. This can include even cases in which one wants to characterize the behavior of potential competitiors by assigning them potential β_j efficiency values with corresponding probabilities of occurrence – which can then be tested by positioning each DMU_j in the functional in order to determine whether its probability of exceeding the specified β_o is sufficiently high to warrant revisions of the value that was initially hypothesized.

There is more research to be undertaken, of course, and this can take a variety of forms. Extensions to more general classes of distributions and decision rules represent one set of possible research avenues. Another set would extend the analyses to other types of DEA models and CCP programming characterizations. Nor does this end the possibilities. As a case in point, we might note that the above models all have been accorded what might be called the "intersection form" of CCP. Another direction of research could proceed from what might be called the "union form" of CCP which, using the notation in Wagner (1969, p. 670ff.), can be adjusted to our type of DEA analysis by writing

$$
P\left(\frac{u^T \tilde{y}_1}{v^T \tilde{x}_1} \leq \beta_1, \dots, \frac{u^T \tilde{y}_n}{v^T \tilde{x}_n} \leq \beta_n \right) \geq \alpha
$$

in place of (4). One needs to be aware of difficulties like those described by Wagner, of course, but these lines of research nevertheless offer important possibilities, and a start toward exploiting these types of joint chance constraint for probabilistic efficiency evaluations to be conducted simulaneously over a collection of DMUs is offered along lines like those set forth in Cooper et al. (1996).

Acknowledgements

The authors are grateful to Sten Thore and R.M. Thrall for helpful comments. Support from the IC[2] Institute of the University of Texas at Austin is gratefully acknowledged.

References

Banker, R.D., Maximum likelihood, consistency and DEA: Statistical foundations, Management Science 39, 1993, 1265–1273.

Banker, R.D., Stochastic data envelopment analysis, Working Paper, Carnegie-Mellon University School of Urban and Public Affairs, Pittsburgh, PA, 1986.

Banker, R.D. and W.W. Cooper, Validation and generalization of DEA and its uses, TOP 2, 1994, 249–296.

Banker, R.D., A. Charnes and W.W. Cooper, Some models for estimating technical and scale inefficiencies in data envelopment analysis, Management Science 30, 1984, 1078–1092.

Banker, R.D., A. Charnes, W.W. Cooper, J. Swarts and D.A. Thomas, An introduction to data envelopment analysis with some of its models and their uses, Research in Governemental and Nonprofit Accounting 5, 1989, 125–163.

Charnes, A. and W.W. Cooper, Deterministic equivalents for optimizing and satisficing under chance constraints, Operations Research 11, 1963, 18–39.

Charnes, A. and W.W. Cooper, *Management Models and Industrial Applications of Linear Programming*, Wiley, New York, 1961

Charnes, A. and W.W. Cooper, Goal programming and multiple objective optimizations, European Journal of Operational Research 1, 1977, 39–54.

Charnes, A. and W.W. Cooper, Preface to topics in data envelopment analysis, Annals of Operations Research 2, 1985, 59–94.

Charnes, A. and W. W. Cooper, Programming with linear fractional functionals, Naval Research Logistics Quarterly 9, 1962, 181–185.

Charnes, A., W.W. Cooper, J.K. DeVoe and D.B. Learner, Demon, Mark II: An extremal equation approach to new product marketing, Management Science 14, 1968, 513–524.

Charnes, A., W.W. Cooper and J. Hsu, A chance constrained programming approach to multiple-objective sample designs in auditing, Working Paper, University of Texas Graduate School of Business, Austin, TX, 1986.

Charnes, A., W.W. Cooper and E. Rhodes, Evaluating program and managerial efficiency: An application of data envelopment analysis to program follow through, Management Science 27, 1981, 668–697.

Charnes, A., W.W. Cooper and E. Rhodes, Measuring the efficiency of decision making units, European Journal of Operational Research 2, 1978, 429–444.

Charnes, A., W.W. Cooper and G.H. Symonds, Cost horizons and certainty equivalents: An approach to stochastic programming of heating oil, Management Science 4, 1958, 235–236.

Cooper, W.W., Z.M. Huang and S.X. Li, Chance constrained programming formulations for stochastic charaterizations of efficiency and dominance in DEA, Journal of Productivity Analysis (submitted, 1996).

Cooper, W.W. and Y. Ijiri (eds.), *Kohler's Dictionary for Accountants*, 6th ed., Prentice-Hall, Englewood Cliffs, NJ, 1983.

Desai, A.A. and A.P. Schinnar, Stochastic data envelopment analysis, Working Paper, The Ohio State University, 1987.

Jagannathan, R., Use of sample information in stochastic recourse and chance constrained programming, Management Science 31, 1985, 96–108.

Kahane, Y., Determination of the product mix and the business policy of an insurance company – a portfolio approach, Management Science 23, 1977, 1060–1069.

Land, K.C., C.A.K. Lovell and S. Thore, Chance-constrained data envelopment analysis, Managerial and Decision Economics 14, 1993, 541–554.

Land, K., C.A.K. Lovell and S. Thore, Productive efficiency under capitalism and state socialism: An empirical inquiry using chance-constrained data envelopment analysis, Technological Forecasting and Social Change 46, 1994, 139–152.

Land, K., C.A.K. Lovell and S. Thore, Productive efficiency under capitalism and state socialism: The chance-constrained programming approach, *Public Finance in a World of Transition, Proceedings of the 47th Congress of the International Institute of Public Finance*, P. Pestieau, ed., 1992, supplement to Public Finance/Finances Publique 47, 1992, 109–121.

Leibenstein, H., *Beyond Economic Man*, Harvard University Press, Cambridge, 1976.

Rhodes, E., Data envelopment analysis and related approaches for measuring the efficiency of decision making units with an spplication to program follow through in U.S. public school education, Ph.D. Thesis, Carnegie Mellon University School of Urban and Public Affairs, Pittsburgh, PA, 1978. Also available from Michigan University Microfilms, Inc., Ann Arbor.

Seiford, L.M. and R.M. Thrall, Recent developments in DEA: The mathematical programming approach to frontier analysis, Journal of Econometrics 46, 1990, 7–38.

Sengupta, J.K., Data envelopment analysis for efficiency measurement in the stochastic case, Computers and Operations Research 14, 1987, 117–129.

Sengupta, J.K., Data envelopment with maximum correlation, International Journal of Systems Science 20, 1989, 2085–2093.

Sengupta, J.K., Efficiency measurement in stochastic input-output systems, International Journal of Systems Science 13, 1982, 273–287.

Sengupta, J.K., Robust efficiency measures in a stochastic efficiency, International Journal of Systems Science 19, 1988, 779–791.

Sharpe, W.F., A simplified model for portfolio analysis, Management Science 9, 1963, 277–293.

Simon, H.A., *Models of Man*, Wiley, New York, 1957.

Stedry, A.C., *Budget Control and Cost Behavior*, Prentice-Hall, Englewood Cliffs, NJ, 1960.

Stigler, G.J., The Xistence of X-efficiency, American Economic Review 66, 1976, 213–216.

Wagner, H.M., *Principles of Operations Research*, Prentice-Hall, Englewood Cliffs, NJ, 1969.