C. I. P. Working Paper No. 342
September 8, 1977

INDIVIDUAL DIFFERENCES IN SOLVING PHYSICS PROBLEMS[1].

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Children from age six onwards spend a large part of their lives in elementary schools, learning school subjects whose formal structure is not very different from that of the subjects they will study in high school and college. Hence, it is perhaps not inappropriate to report at this conference some research on individual differences in adult competence in a subject, elementary physics, that is typically taught in high school and college. An understanding of the bases of adult competence may cast light on the skills the child must attain enroute to such competence, in particular, prerequisite skills in arithmetic, reading, and algebra. Moreover, the method of analysis used here could also be used in studying the knowledge demands of elementary school subjects.

We shall follow the strategy used in other recent work in information processing psychology (e.g., Klahr & Wallace, 1976) of trying to establish what is learned before investigating how it is learned -- that is, of studying performance first, and then learning. The task domain we shall examine is a part of elementary physics. More specifically, we shall be concerned with the topic of motion in a straight line, which

[1] This research was supported by Research Grant MH-07722 from the National Institute of Mental Health.

occupies about one week of a one-year high school or 1st-year college physics course. Still more specifically, we shall use the treatment of this topic found in a widely used high school physics textbook that employs algebra but no calculus (Taffel, 1973). We are interested in what it is that a student knows when he has mastered the chapter, and how he makes use of this knowledge when he goes about solving problems. Our method of study has been to gather problem-solving protocols from two subjects: (1) a subject with a strong mathematical background and wide experience in solving problems of this kind, and (2) a subject who had taken a single course in physics many years previously, and who had an adequate, but not unduly strong, background in algebra. From the comparison of the behavior of the more experienced and the less experienced subject, we shall seek to draw some conclusions about the learning process, and to comment on which part of the skill is least readily learned by standard methods of studying textbooks and working examples.

A small number of analyses of school subjects appear in the literature, but not always at the level of detail proposed here (and which we think essential for our purposes). Gagne (1963) pioneered in this kind of analysis with his studies of elementary arithmetic skills. Paige & Simon (1966; see also Bobrow, 1968) studied the processes involved in understanding and solving algebra word problems. Novak (1976) simulated the processes for solving physics problems in statics (levers). Marples (1974) has analysed the logical structure of problems in mechanics and electricity. Bundy (1975) and Bundy, et al. (1977) have analyzed student performance in solving pulley problems; Larkin (1976) has analyzed simple dynamics problems. Greeno (1976, 1977) has analyzed the performance of high school students solving several classes of geometry problems and has constructed a program that closely simulates their performance. Brown and Burton (1975) have constructed a computer program for tutoring students learning to solve electronics circuit problems. Bhaskar & Simon (1977) constructed a program for generating problems in chemical engineering thermodynamics, and Bhaskar (1977) has analyzed human problem solving behavior in chemical engineering, cost accounting, and business policy cases.

The Task Content

Motion in a straight line is taken up in the fifth chapter of Taffel (out of 32), and occupies 22 pages (out of about 550). Previous chapters have already introduced methods and units for measuring length, time, mass and weight (Chapters 2 and 3), and the concepts of vector, force, and motion (Chapter 4), but not acceleration. The fifth chapter introduces the notions of uniform and accelerated motion, units of acceleration, average and instantaneous speed, relative speed, motion at constant speed, and uniformly accelerated motion. The gravitational constants in English and metric systems ($32ft/sec^2$ and $9.8m/sec^2$, respectively) are also explained. The following formulas are given, together with derivations and verbal explanations.

(1) $S = v^*T$,

where $\underline{S}$ is distance, $\underline{v}^*$ is average speed, and $\underline{T}$ is time.

(2) $S = vT$, for constant speed, $\underline{v}$.

(3) $a = (v-v_0)/T$,

where $\underline{a}$ is acceleration, $\underline{v}$ is final speed, and $\underline{v_0}$ is initial speed.

(4) $v = v_0 + aT$

(5) $v^* = (v_0 + v)/2$

(6) $v^* = v_0 + 1/2(aT)$

(7) $S = v_0T + 1/2(aT^2)$

(8) $v^2 - v_0^2 = 2aS$

Equations are also given for the special case where the body starts from rest, i.e., where $v_0 = 0$:

(9) $v = aT$

(10) $S = 1/2(aT^2)$

(11) $v^2 = 2aS$.

Note that the symbol v is used both for constant speed and for final speed in uniformly accelerated motion. In other respects, the symbolism is unambiguous. The English-language text just preceding the introduction of each equation always specifies the condition (e.g., constant acceleration, constant speed) under which the equation holds.

The material we have just described occupies the second through the tenth pages of the chapter; the remainder of the chapter, which we shall not consider, is devoted to relative motion, graphical analysis of motion, and motion of falling bodies. The text and formulas are illustrated by seven sample problems, worked out step by step. These are followed by 15 "questions" and 25 "problems." The two subjects whose behavior we shall analyze read the text, answered the questions, and worked the problems in order.

The whole empirical and formal content of this chapter is modest, for it can be summed up in the 11 equations given above. Even these are not independent, for they can all be derived from the three relations: (1) $S = v^*T$, (4) $v = v_0+AT$, and (5) $v^* = (v_0+v)/2$. (To solve the problems, the student must also know the value of g, the gravitational constant.) Without specifying precisely what a "thing" is, we may say that mastery of this chapter requires the student to learn not more than about ten "things." If this chapter is typical of the whole text, then a one-year physics course calls for the mastery of about 300 "things." Again, if this course is typical of high school courses, a student carrying four courses might be expected, during a school year, to learn more than 1,000 but less than 2,000 "things." We may compare these crude estimates with the estimate (Chase & Simon, 1973) that a chess master spends ten years or more learning about 50,000 chess patterns which he can recognize on a chess board. By way of further comparison, a Japanese elementary school child learns to read and write two to three hundred ideograms each year, and to associate with each the meanings and "readings" (pronunciations) it can have in different contexts.

Solution Times and Paths

As mentioned earlier, our data were obtained from two subjects, one quite expert in working problems of this kind, although without recent practice, the other having fair skill in algebra, but essentially new to the subject of kinematics. Our expert subject, S1, and our novice, S2, each worked the 25 problems at the end of Chapter 5 under standard thinking-aloud instructions, using paper and pencil freely. S2 referred to the textbook when she needed to find or recall a relevant equation, but she made fewer and fewer such references as she proceeded, and none at all in handling the last six or eight problems. S1 did not refer to the textbook. Their protocols for 19 of the 25 problems are analyzed here. Problems 1, 2, 4, 14, and 15, which dealt with relative motion, are omitted, as is Problem 22, which was more complex than the others (involving a pair of moving bodies, instead of only one);that problem will be discussed in a later section.

<u>Times and Protocol Lengths</u>

Table 1 shows, for each subject, the total number of words in each protocol, the time in seconds required to solve each problem, and the average rate of speech in words per minute. It can be seen that S1 solved some of the problems in less than half a minute, and required only 2.5 minutes for the hardest (Problem 16). Only four of these problems took him more than a minute. There was a small

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Insert Table 1 about here

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upward trend in his solution times from beginning to end of the sequence, but it must be remembered that the subjects solved the problems in the order in which they appeared in the textbook, and they had been arranged by the author so as to increase in difficulty. Hence, difficulty is confounded with practice. S2, the novice, took, on average, about four times as long per problem as S1. Her solution times ranged from about one minute to over 9 minutes (Problem 24). While all of S1's solutions were correct, S2 arrived at incorrect answers for two problems, both of which involved arithmetic errors.

S1 produced, on average, about 160 words per minute. His average rate of speech was slightly higher on the easy problems, and slightly lower on the hard ones. After allowance is made for differences in problem difficulty, there appears to be no trend in his rate of verbalization over the course of the problem-solving session. S2's speech was less than half as fast as S1's, averaging about 70 words per minute, and the number of words in her protocols averaged about twice the number in S1's.

Characteristics of the Protocols

Figures 1 and 2 about here

Figures 1 and 2 exhibit the protocols of these subjects on Problem 19. These protocols illustrate quite clearly the characteristic differences between the verbalizations of the two subjects, and can be regarded as reasonably typical of the other problems in the set. First, however, the similarities: Both subjects begin by reading the problem aloud, as they were instructed to do. They then evoke (or sometimes, in the case of S2, look up) appropriate equations to describe the physics of the situation, and then solve the equations to find the unknown quantitites. They answer the two questions -- about average speed and duration -- in the order in which they were asked.

There are also important differences between the two protocols, the most obvious being the typical two-to-one ratio of their lengths. After reading through the first question (Lines 1-4), S1 simply calculates the average speed (Line 5), without mentioning the formula he has evoked from memory and is applying. He then reads the second question (Line 6), and again carries out the calculation without mentioning the formula ($v^* = S/T$) he is using. He does mention the values of the givens (Lines 7-8), and then repeats them (Lines 9-11) before carrying out the computation successfully (Lines 12-14). Even though the formulas are not mentioned explicitly, it is entirely clear from the computations that they are Equation 5 (specialized for $v_0 = 0$) and Equation 1 (transformed to solve for $\underline{T}$ in terms of $\underline{S}$ and $\underline{v*}$). S2, on the other hand, writes down the first given (Line 2) as she reads the problem (Lines 1-5), making a correction in a reading error as she goes. She then states explicitly the formula she is going to use (Lines 6-8), and applies it (Line 9). Next, she checks her result (Lines 10-17), in particular, the correctness of the formula she used. Now she interrupts herself (line 18) to comment on a possible mistake in the previous problem. She reads the second question (Line 19), repeats it (Line 20), evokes an appropriate formula (Lines21-22), transforms the formula to make I the dependent variable (Line 23), substitutes numerical values for the givens in the formula (Lines 24-25), and carries out the numerical computation (Lines 26-27). (The answer, although incorrect as verbalized, is correct on the worksheet.)

While a single verbalization covers S1's evoking a formula, putting it in the appropriate form, and substituting the numerical values for the independent variables, each of these three steps is verbalized separately by S2. This may merely reflect a difference in the subjects' programs of verbalization. On the other hand, it may represent an automation of successive steps in S1's program that results in the individual steps' being no longer available for verbalization. If the latter interpretation is adopted, then S1's program calls for combining information from the problem statement with knowledge of physical laws at the earliest possible moment, by instantiating the laws with the given values of variables as soon as the former are evoked from long-term memory. Instantiation appears to be less automatic in S2's program: more time elapses before information from the two different sources are brought together, and in some protocols, formulas are produced in literal form without being immediately instantiated.

In S1's protocols, almost all of the verbalizations are directly descriptive of the calculations he is making. There are relatively few verbalizations of plans, or other "meta-statements" about the process. These occur mainly in protocols for the problems that took him 50 seconds or more. In Problem 19, for example, S1's only meta-comment is "wait a while" (Line 9). S2, on the other hand, makes a number of such comments: "That's got to be right" (Line 12), "Why divided by 2? The average of the two speeds, right?" (Lines 15-16), "There's something wrong with that" (Line 18), and "So let's worry about that" (Line 26). In the last six problems (Problems 19-21, 23-25) S2 averages about five meta-comments per problem, while S1 averages only one per problem.

The contents of the two subjects' meta-comments are similar: observations that a mistake has been made; a comment on the physical meaning of an equation; the question, "What do we know?" (i.e., what are the givens, or what formulas are available?); statements of plan or intent (e.g., "Let's just clear fractions."), evaluations (e.g., "Is that right?"); and a few others. S2 expresses uncertainty fairly often about the steps or calculations she has taken, S1 very rarely.

Solution Paths

We observed in the previous section that the basic structures of the two protocols are quite similar. Immediately after reading the problem, they evoke an equation from long-term memory. The equation is instantiated by substituting in it quantities that are given in the problem statement, and then solved. This process is repeated (usually two or three times) until values have been found for the unknowns mentioned in the problem statement. This is the same basic structure as reported by Bhaskar & Simon (1977) in their studies of a subject solving thermodynamics problems and Marples (1974) in studies of subjects solving physics problems. The fundamental cycle can be described as:

FIND EQUATION
INSTANTIATE EQUATION
SOLVE EQUATION

The scheme may be applied repetitively or recursively -- that is, if values for all the independent variables in an equation are not known, a new equation may be found in one of the unknowns, instantiated, and solved.

Although both subjects used this scheme, they often retrieved different equations from long-term memory to find the same unknowns. The 19 problems called for the values of 32 unknowns to be found. In 19 of these 32 cases, S1 and S2 used essentially the same sequence of equations (solution path) to solve for an unknown; in 13 cases, they used different paths (see Table 2, Columns 3 and 5).

Insert Table 2 about here

The most frequent difference (9 cases) was that S1 used Equation 5 followed by Equation 1 to solve for S, while S2 used Equation 7 or its specialization, Equation 10. The converse difference did not occur even once. Nor did S1 on any occasion use Equation 7, although he did use Equation 10 in five problems. S2, on the other hand, used Equation 7 eight times and Equation 10 seven times.[2].

The other principal difference between the two subjects (3 cases) was that S2 used Equations 8 or 11, while S1 did not. In fact, S1 reported that he was unfamiliar with Equations 8 and 11, and he was willing to accept them as correct after he had rederived them. Notice that these equations have no obvious direct physical interpretation.

[2] Table 2 shows only successful solution paths. S2, especially, actuallymade a number of false starts and corrections in solving most of the problems. However, the aim we set ourselves in this paper was to study the knowledge demands made on the student in solving this type of physic problem. The analysis in this and the following section reflects the way each of the subjects met these demands.

A Production System

A rather close simulation of the behavior of both subjects -- in particular, of their successful solution paths -- can be obtained within the framework of a common program structure. The simulation attempts to account for the sequence in which required equations are retrieved from long-term memory and solved. The differences between S1 and S2 are to be explained by rather modest differences in the way in which equations are cued by information in the problem and retrieved from long-term memory.

The simulations take the form of simple production systems. A production system (Newell & Simon, 1972) is a program consisting of an ordered set, or list, of productions. Each production consists of a condition part and an action part. The rules for the operation of the system are these: (1) the productions are arranged in linear order, and the conditions of each one are tested in turn; (2) if, upon testing, the conditions of a production are found to be satisfied, the action part of that production is executed; (3) after execution of a production, the testing process resumes, beginning with the first production on the list.

In the production systems under consideration, the conditions are the presence or absence of particular variables in an equation. Associated with each of the kinematics formulas is the name of its dependent variable, and the list of its independent variables. As each problem is being solved, lists are kept, for that problem, of the variables whose values are known and of the variables whose values are wanted. Tests can then be performed to determine whether the values of any of the independent or dependent variables in the formula are known, or whether they are wanted. Clearly, a particular formula can be solved only if the values of all independent variables are known; generally, there is a reason to solve the formula only if the value of the dependent variable is wanted. The tests used in the production systems for S1 and S2 are based on these considerations. To be specific:

1. The condition for S1's productions is that the values of all independent variables be known. If they are, the action part of the production is executed -- that is, the equation is solved for the dependent variable.

2. The conditions for S2's productions are that: (1) the dependent variable of the production's formula be on the "wanted" list; and (2) the values of all its independent variables be known. If both conditions are met, the action part of the production is executed. If the first condition is met, but not the second, then the name of the first independent variable whose value is unknown is placed on the "wanted list", but the production is not executed.

The production systems for S1 and S2 also differ in having different orderings of the productions. In all other respects they are identical. The lists of productions of the two systems are shown in Figures 3 and 4.

The production systems are even simpler than the lists suggest. In S1's production system, only four distinct equations appear: Equations 1, 4, 5, and 7 of our original set, together with the equations obtained by permuting independent and dependent variables in these. Thus, P1, P4, and P6 correspond to Equation 4; P2, P3, and P7 to Equation 1; P5 to Equation 5; and P8, P9, and P10 to Equation 7. They are listed separately for simplicity in writing the system, and no psychological significance should be attached to this format. Moreover, S1 actually only used Equation 10 instead of the more general Equation 7, but P8, P9, and P10 have been kept in the general form to avoid having to express the condition, $v_0 = 0$ in the production system.

In the same way, S2's twelve productions correspond to only five distinct equations: P1, P5, and P12 to Equation 1; P2, P6, and P11 to Equation 7; P3 and P8 to Equation 8; P4, P7, and P9 to Equation 4; and P10 to Equation 5. P3 and P8 appeared in her protocols only in the special case of Equation 11 (i.e., the special case where $v_0 = 0$).

Table 2 compares the subjects' paths with the paths used by the simulation programs on each of the 19 problems. There is an extremely close correspondence. S1 uses Equation 10 instead of Equations 5 and 1 to solve for S in Problems 5, 11, and 12. (These are the only problems involving both vertical motion and $v_0 = 0$, which may be the cues that divert S1 to this path.) In Problems 23 and 25, the simulation, but not S1, computes the value of a, which is not called for by the problem statement. In Problem 24, S1 performed complex manipulations using Equations 10 and 9 to find the time, and then applied Equation 9, as did his simulation, to find the acceleration. (See pp. 17-18 for a fuller discussion of his solution of this problem.) In Problem 13, S1 solves for a and v first, then for v^*, while the simulation reverses the order (thereby following the order in which they are asked in the problem text). In this problem, S1 also uses v*5 instead of v*1, which can be found directly from the givens. In the case of S2, the match between simulation program and human protocol is even closer, the paths differing only for Problems 18, 19 and 25. In Problem 18, S2 uses Equations 5 and 1, instead of her customary Equation 10, to solve for S. In Problem 19, she solves first for v^*, then uses Equation 1 to solve for T, while the simulation solves successively for A, T, and v^*.

In Problem 25 she also solves first for V^* and then applies S1; the simulation first finds A so that it can use Equation 7 to find S. The only other difference is that in Problem 16 the simulation reversed the requested order in finding the dependent variables.

The production systems, therefore, appear to capture very well the processes the two subjects are using to solve these 19 problems. What do they tell us about the nature of skill and expertness? There are two ways to characterize the differences between the two systems. First, S1's system represents a "working forward" strategy while S2's represents a "working backward" strategy. That is to say, S1 operates from the givens in the problem, solving successively the equations that can be solved with these givens, without much attention at the outset to the particular variables that the problem statement asks him to evaluate. Only in Problem 16, which took him the longest time to solve, and Problem 24, the second longest, does S1 make any comments that can be interpreted as means-ends analysis. In the course of solving Problem 16, he remarks, "no, what am I doing. I'm finding . . ah . . " . . . "no, I don't want the T; where are we now?" . . . "So we have to find the time first." And while solving Problem 24 he says, "and what do we know? What we know is the final velocity." S2, on the other hand, evokes equations in which the desired quantities are dependent variables, and if not all the independent variables in these are known, sets up subgoals to solve for them.

Viewed in this way, S2's behavior seems more goal directed than S1's -- at first blush a surprising result. However, this phenomenon has also been observed in subjects working thermodynamics problems (Bhaskar & Simon, 1977; see also Marples, 1974). When the problem is very easy, the expert knows that he can solve it simply by solving equations as he comes to them, so to speak. When the problem is harder, his behavior becomes more purposeful and is guided by a means-ends analysis of the goals he is seeking to reach. Thus, the more "primitive" approach of S1 is to be attributed to his confidence that forward search will lead quite directly to a solution of the problem, and will not generate a large and inefficient search. This confidence is based on his experience with the problem domain.

A Comment on Physical Intuition

Physicists and engineers often refer to "physical intuition" as an essential component of skill in solving physics problems. Sometimes, solving a problem with the help of physical intuition is contrasted with solving it "simply by plugging in the formulas." The facts that the idea of physical intuition is somewhat elusive and that it has not been defined operationally do not mean that the phenomenon underlying it is unimportant to problem solving skill. We should like to venture here an interpretation of physical intuition in information processing terms, and provide some evidence that S1 made important use of it.

Physical intuition might be interpreted in the following way: When a physical situation is described in words, a person may construct a perspicuous representation of that situation in memory. By a perspicuous representation, we mean one that represents explicitly the main direct connections, especially causal connections, of the components of the situation. For example, in a statics problem involving a ladder leaning against a wall, the representation might be an associational structure with nodes for the ladder, the wall, the floor, and the points of contact between the ladder and the wall and the ladder and the floor. The force of gravity acting on the ladder would be associated with the ladder, and the forces at the points of contact would be associated with those points. Once this schema had been constructed in memory, it would be a straightforward matter to construct the equations of equilibrium for the situation. In fact, Novak (1976) has built a computer program for understanding statics problems stated in natural language that proceeds in exactly this way: first it constructs a schema representing the essential relations in the situation, then it sets up equations corresponding to this representation. In our present terminology, we would say that the program exhibits physical intuition.

We claim that S1 used physical intuition in solving our kinematics problems. That is, he first translated the English prose of the problem statements into physical representations, then used those representations to select and instantiate the appropriate equations. The representations reflected his causal view of uniformly accelerated motion, a view that can be summed up in two statements: (1) a distance is traversed in uniform motion by the cumulation of equal unit distances incremented over successive unit time intervals; and (2) a velocity is acquired, in uniform acceleration, by the cumulation of equal unit velocities incremented over successive unit time intervals. In this representation, velocity is measured by the unit distances traversed in unit times of statement (1), while acceleration is measured by the unit velocities of statement (2).[3]

What is the empirical basis for claiming that S1 used a physical representation of

[3] We have modeled these two statements on Galileo's definitions of uniform motion ("one in which the distances traversed by the moving particle during any equal intervals of time, are themselves equal.") and uniform acceleration ("A motion is said to be uniformly accelerated, when starting from rest, it acquires, during equal time-intervals, equal increments of speed.") in Dialogues Concerning Two New Sciences, translated by Henry Crew and Alfonso de Salvio, Evanston: Northwestern University Press, 1939, pp. 154 and 162, respectively. Since Galileo was deriving, for the first time, the kinematic laws of uniform motion and acceleration, he was striving in these pages to infer a mathematical description from a physical one; hence, these definitions and the passages surrounding them may plausibly be taken as indicating Galileo's physical representation of the situation. Galileo uses this physical representation to derive the equations that we have labeled Equations 2 and 9. He then proceeds, still working from the physical representation, to derive Equation 1, where average velocity is given by the definitional Equation 5 (ibid., pp. 173-174), and finally (ibid., p. 174-175) the celebrated Equation 10.

the sort just described, instead of going directly from the problem statements to the equations? The evidence is far from conclusive, and we will have to let the reader decide whether he finds it persuasive. First, we have the fact, already noted, that S1 generally calculates distance from Equation 1 rather than Equation 10, even though this choice requires him first to solve Equation 5, and sometimes Equation 9. To be sure, S1 does use Equation 10 in Problems 5, 11, 12, 13, and 16, but in three of these five cases (Problems 5, 13, and 16), he is not satisfied with his answer until he checks it, or tries to check it, using the other path. Thus, in Problem 5, after using Equation 10 to find the distance traversed by a rolling ball, he says: "That seems like a lot .. ah, oh, in 4 seconds, sure, its final velocity was 12 meters per second so half of that is 6 meters per second and 4 seconds is 24. So that figures." Here he uses the known final velocity to find the average velocity, and the average velocity and the known time to check the distance. In Problem 11, S1 actually begins by using Equation 9 to find final velocity from time and acceleration, but then (for no reason that can be discerned from the protocol) shifts to Equation 10. He does not, however, check his answer. In Problem 13, after using Equation 10 to find the acceleration from the time and distance, then the final velocity and average velocity (4 meters per second), he concludes with: "I should have known that since it went down 12 meters in 3 seconds." In Problem 16, the distance and acceleration are given, and the time is called for. After using Equation 10 to solve the problem, S1 says, "Another way to do that would have been to say it goes down 88.2 meters . . oh, no, I couldn't do that without first solving for the time, so that's ok." That is, he tries to check his calculation with Equation 1, but discovers that neither the time nor the average velocity is given. Problem 24 provides a striking example of checking. This is the only problem in which distance and terminal velocity are given, while acceleration and time are to be found. S1 first evokes Equation 9, to solve for A, but realizes that T is not given ("oh, no, that's not quite as easy as that"). He then evokes Equation 10, but again observes that he has two unknowns. Reviewing what is given, he notes that he can eliminate A from Equation 10 by using Equation 9. He thereby obtains $s = 1/2\ vT$, pauses, and says, "oh, of course. The distance is one half times the velocity -- the terminal velocity -- times the time."

There are no problems in which S1 proceeds in the opposite direction -- that is, in which he uses Equation 1 to find the distance, and then checks his result with Equation 10. We conclude that the former equation has some priority over the latter, and we attribute this priority to the fact that Equations 1 and 9 (and Equation 4, which is the generalization of the latter) derive directly from the hypothesized physical representation. We would also (and still more speculatively) attribute S1's assurance that he is using the correct equations to the fact that he is not simply recalling them from memory, but is either generating them from the physical representation, or at least using the latter to check his recall. When S1 uses an equation that is not based directly on the physical representation (Equation 10), he exhibits no such assurance, and usually feels obliged to check his result.

There are only two comparable paragraphs in S2's protocols, in spite of her much more frequent use of Equations 7 and 10, where she checked using the other path. These occur in Problems 5 and 17. In Problem 18, she does the reverse: she uses Equation 10 to check a result derived by the more intuitive path. Hence, there is little evidence that S2 used a physical representation as the source of her equations or to check her results.

Our confidence in this explanation of S1's superior performance is buttressed by evidence for the use of such physical representations in the literature. In particular, Paige & Simon (1966), who presented their subjects with algebra problems that corresponded to physically unrealizable situations, found that many of the subjects unintentionally transformed the problems into similar but physically realizable forms.

A More Difficult Problem

The 22nd problem at the end of Taffel's Chapter 5 is one of the more difficult ones. It reads:

> At the moment car A is starting from rest and accelerating at $4m/sec^2$, car B passes it, moving at a constant speed of 28m/sec. How long will it take car A to catch up with car B?

The problem refers to two moments in time, which we will call t_0 and t_1. T_0 is the time when car A starts from rest, just as car B passes it; t_1 is the time when car A catches up with car B. During the interval $T = t_1 - t_0$, the two cars travel the distance, S. That the two cars travel the same distance, S, in the same time, T, must be inferred by the problem solver from the language of the problem statement.

There are a number of ways to solve this problem, three of which are reasonably direct:

(1) Remembering that T and S are the same for both cars, we have from the first sentence, together with Equation 10, $S = 1/2(4T^2)$. From the second sentence, together with Equation 2, we have, $S = 28T$. Eliminating S between these two equations, we solve $2T^2 = 28T$ for T, obtaining $T = 14$.

(2) Starting with the two equations from the previous solution, we solve the

second for T = S/28, and substitute this value in the first, obtaining $S = 2(S/28)^2$. Substituting the solution, S = 392, in the second equation, we again obtain T = 14.

(3) The average speed, v^*, of car A over T must be the same as the average speed of car B, which is 28. But $v^* = (v_0 + v)/2$. Since $v_0 = 0$, it follows that the terminal velocity of car A is twice its average velocity, or 56. Using the equation v = 56 = 4T, we immediately obtain T = 14. This third path is the one that rests most directly on physical intuition, as we have defined that term.

Performance of S1 on Problem 22

The experienced subject used method (1) to solve Problem 22. His protocol is so brief we quote it in full:

> [Reads problem]
>
> 1. Ah, well, that's a little trickier.
> 2. Ah, Car A goes a distance of $1/2\ aT^2$
> 3. where acceleration is 4 meters per second.
> 4. So it's . . . ah . . . 2 T^2 is the distance it goes,
> 5. And the other car goes a distance of 28 meters per second times T,
> 6. so 28 times T --
> 7. and so $2T^2 = 28T$
> 8. Which says that T = 14.
> 9. what . . . seconds, I guess.
> 10. Ah, so we will assume that that will catch up in 14 seconds.
> 11. Now, let's see if that makes any sense.
> 12. In 14 seconds that car would be . . ah . . . going at a velocity of
> 13. . . . ah . . . whatever 14 times 4 is
> 14. which is 56.
> 15. So it would have gone . . . um . . . ah . . . at an average velocity of 1/2 that
> 16. or 28,
> 17. which is right to catch up with the other one.

Thus, Subject 1 spoke 84 words while solving the problem, by the first of the three solution paths described above. He then spoke 56 words while checking his result by the third solution path. Although he used Equation 10 in solving the problem, we see that he checked his result by the third path, giving us additional evidence for his reliance on a physical representation. The protocol provides no clues as to how he chose his steps. In lines 2 through 4 he simply translates the facts about Car A into an equation, and in lines 5 and 6, the facts about Car B. He then equates the two distances, in line 7, and solves the resulting equation.

On the basis of the information he uses, but without explicit support from the language of the protocol, we can infer that he must have used processes such as the following:

(a) He generated some kind of problem representation that incorporated the starting time, t_0, and location, s_0, and the terminal time, t_1, and location, s_1, thus defining the time interval, T, and distance, S.

(b) Mention of the constant acceleration of car A evoked from his long-term memory Equation 10, $S = 1/2(aT^2)$.

(c) Similarly, mention of the constant speed of car B evoked from his long-term memory Equation 2, $S = vT$.

(d) He noticed that both of the equations had the same dependent variable, S, and he set the two equations equal.

(e) He noticed that the new equation contained only the single variable T, and he solved for it.

In this whole sequence, perhaps the most sophisticated processes are those involved in interpreting key terms in the original problem statement. The first sentence of the text is of the form:

At the moment X, Y,

where X is a pair of events involving car A, and Y a pair of events involving car B. The sentence asserts that all of these events took place at one time, t_0: $v(A,t_0) = 0$, $a(A,t_0) = 4$, $v(B,t_0) = 28$, and B passes A. This last condition may be expressed algebraically by the assertion that cars A and B are in the same location at time t_0: $s(A,t_0) = s(B,t_0) = s_0$. The "how long" of the second sentence implies a time interval, T, and hence a second point in time, t_1, at which car A "catches up" with car B. The parser must be clever enough to know that the latter phrase means that the two cars have the same location at that time: $s(A,t_1) = s(B,t_1)$. Furthermore, from the fact that cars A and B have the same locations at times t_0 and t_1, he must infer that they have gone the same distance, $S = s_1 - s_0$, during the time interval, $T = t_1 - t_0$. Once these translations had been accomplished, a fairly straightforward set of "noticing" processes would be capable of evoking the appropriate equations.

S1's processing scheme for Problem 22 involves only a slight elaboration of the scheme he used in solving the simpler problems. The main components of this scheme are (1) parsing capabilities powerful enough to handle such phrases as "at the moment," and "catch up"; (2) capabilities for creating a semantic representation of a physical situation, and drawing inferences (e.g., that the times and distances are equal for the two cars); (3) capabilities for evoking physical relations (equations) from long-term memory, cued by suitable words or phrases in the text; and (4) capabilities for solving simultaneous equations. Again, there is no evidence in the protocol that the subject planned the path he took -- and the brevity of the protocol argues against any elaborate planning effort. A simple set of productions could produce such a path without planning.

Performance of S2 on Problem 22

S2's protocol for Problem 22 extends over 18 minutes until the problem is solved, and contains about 1200 words, which were encoded into some 150 statements. The protocol divides neatly into five major episodes. The first episode (lines 0-42), which took about 6:15 minutes, was occupied with reading the problem, translating its content into algebraic equations, making some inferences from the semantic representation of the problem, and evoking some physics equations from LTM. The second episode (lines 43-78), which took about 3:30 minutes, is occupied with rereading the problem statement, and summarizing the information that was generated in the previous episode. No new information is produced in the second episode. The third episode (lines 79-109, 3:30 minutes) begins with S2 setting some fairly specific goals, and then pursuing them. Up to this point, she has failed to evoke one of the physics equations essential for solving the problem (Equation 10). This failure is consistent with the production system we have postulated for S2, for the problem statement does not ask for a value for S, the dependent variable in Equation 10. The formula is finally evoked from LTM (line 92), and equations adequate for solving the problem are set up. The equations are judged to be "too complicated," and S2 abandons this plan. The brief fourth episode (lines 110-117, 1:00 minute) involves an unsuccessful effort to execute an infeasible plan. In the fifth, and final, episode (lines 118-150, 4:30 minutes), S2 returns to the plan of the third episode. This time she is not deterred by the complexity of the resulting expressions, and solves the problem.

The second of the three solution paths was employed. Ignoring, for the moment, S2's search activities, the solution path itself is longer than that of the expert subject, S1. For Car A, S2 uses the formula, $S = 1/2(aT^2)$, substituting 4 for a on the basis of the problem statement. For Car B, she uses $S = vT$, where $v=28$ is also given in the problem statement. From the semantics of the problem, S2 has deduced that the <u>S</u>'s and the <u>T</u>'s in the two equations are equal. Next she solves the second equation for $T = S/v$, and substitutes this value in the first, obtaining: $S = 2(S/28)^2$. She then solves this equation for <u>S</u>, and, substituting the value in the second equation, finds <u>T</u>, the desired answer.

As with the simpler problems, S2 is much more meticulous than S1 in mentioning and writing down in algebraic notation all of the facts mentioned in the problem statement (e.g., in lines 2-16 of her protocol). By line 19, however, she has constructed a representation of the situation that permits her to infer that the times and distances are the same for the two cars. In lines 24-32, she deliberately evokes physics formulas that may be relevant to the problem, recalling Equations 2 and 9, but not the crucial Equation 10. These formulas are not evoked, however, as translations of particular statements about cars A and B, but as general laws of physics -- e.g., "We know that distance equals velocity times time," which is true only for the constant velocity of car B, or the unknown average velocity of car A.

These steps have taken S2 more than 6 minutes. She takes the next 3:30 minutes to summarize and recopy this information. In the next 1:30 minutes, S2 arrives at a solution plan, and evokes the missing formula, Equation 10. The rest of the protocol consists of attempts, first unsuccessful then successful, to carry out this plan.

Both subjects, very early in their search, retrieve from memory a physical law to describe the motions of each of the two cars. For S1, the laws are Equations 10 and 2, whose simultaneous solution leads immediately to the solution of the problem.

S2, however, (line 27) retrieves Equation 9, $v = aT$, instead of 10, $S = 1/2(aT^2)$, to describe the motion of car A, and Equations 9 and 2 cannot be solved simultaneously because they have more than two unknowns ($\underline{S}$, $\underline{v}$, and $\underline{v}^*$). S2 notices this almost immediately (line 34), but does not then evoke Equation 10. That equation, which involves the variable $\underline{S}$, is only evoked after S2 notices (line 87) that the problem can be solved as well in terms of $\underline{S}$ as in terms of $\underline{T}$. She then puts $\underline{S}$ on the "want list." The planning episode, lines 79-94, is so crucial to the solution effort that it is reproduced here:

79. Now, we have some things that relate these;
80. we ought to be able to get one in terms of the other.
81. If . . . the . . . We know the velocity
and that seems to be sort of crucial.
82. So, let's see if we can relate those two.
83. So the distance
84. Let's see, no here, time or
85. time equals distance over velocity.
86. And since it's the time we want to find . . .
87. Oh, well, it doesn't really matter.
88. So let's say the distance equals the distance for B,
89. or distance, it doesn't matter for either of them
since they're equal,
90. distance for B equals 28 times the time.
91. And what do we know about the distance in terms of the other?
92. S also equals one half of the acceleration times the time squared
93. or one half of the acceleration, which is 2,
94. times T squared.

The context makes it reasonably clear that the "relate these" of line 79 refers to time and distance. The goal of relating time and distance for car B evokes Equation 2, and the goal of relating them for car A evokes Equation 10. S2 now has all of the information she needs to solve the problem. She is not confident of this, however, for she engages in two brief episodes (lines 95-98 and 110-117) where she attempts to use relations in terms of velocity.

S2 employs the second solution strategy. This strategy follows from the method she uses for solving simultaneous equations: solve the first equation for one variable in terms of the other, then substitute that value in the second equation and solve it. The alternative, setting the right-hand sides of the two equations equal to each other, does not appear to be in her repertory.

Problem 22 is more complex than the others we have considered, because it requires the solution of two simultaneous equations. Nevertheless, the performance of both subjects on this problem, and the differences in their ways of attacking it, are quite similar to their performances and the differences between them on the simpler problems. S2's difficulties on Problem 22 were exacerbated by her failure to evoke Equation 10 promptly, and her temporary abandonment of the correct path because of its complexity. Nearly ten minutes elapsed before a plan of attack was formulated -- a delay related to the fact that S2 did not regard <u>S</u> as a "wanted" variable, and hence did not evoke Equation 10.

<u>Implications for Learning</u>

We can extract from this experiment two kinds of information that have implications for learning. First, we can see if there was any significant change in the behavior of either subject over the sequence of problems. Since S1 was already experienced in these kinds of problems, we would not expect much change in his approach, but we might expect S2's behavior to resemble S1's more closely on the later problems than on the earlier ones. A second implication for learning might be derived from the comparison of the two subjects' styles. What was it that S2 had <u>not</u> learned that would have facilitated her solving the problems? Is there anything we can say about the learning method she used (i.e., studying the chapter of the textbook) that would account for what she had learned and for what she had failed to learn?

Progressive Changes in Behavior

There were no striking changes in the behavior of either subject from the earlier problems to the later ones. This is shown by the fact that a single production system can be written for each subject that predicts behavior quite well for the entire sequence of problems. Nor does S1's advantage in solution times appear to decline over the sequence, as we might expect if S2 were learning rapidly.

On the other hand, S2's learning may be obscured by the fact that the later problems are, on average, more difficult than the earlier ones (for both subjects), and of more varied types. Three of the four problems S1 found most difficult were the only problems of their type (the fourth, Problem 21, was the one where he started with English instead of metric units). These same three problems were also difficult for S2.

There is one clear piece of evidence of learning in S2's behavior, and one that is more speculative. First of all, on a number of the earlier problems, she had to refer back to the textbook to find the appropriate equations, or to verify them. This dependence on the textbook disappeared fairly rapidly. The information was gradually transfered to, and became available from, long term memory. The more subtle change was that, on two occasions, S2 used different methods of solving two problems of the same type. Although Problems 6 and 18 are identical in structure ($\underline{v}_0$, $\underline{a}$, and $\underline{v}$ given, $\underline{T}$ and $\underline{S}$ unknown), S2 uses Equation 10 to solve Problem 6, but finds distance in terms of average velocity in Problem 18. The same shift occurs from Problem 23 to 25 ($\underline{v}_0$, $\underline{v}$, and $\underline{T}$ given, $\underline{S}$ unknown). Hence, in two of the later problems (Problems 18 and 25), S2 uses the path more closely related to physical intuition, where earlier she had used the other path.

We should not be surprised that other clear evidence of learning has not been discovered in the protocols. Even in a restricted task domain, working some two dozen problems does not represent an enormous amount of practice of the requisite skills. In domains of motor skill (playing the piano, riding a bicycle) we would not be astonished if this much practice yielded only modest gains.

Gaps in Skill

No single factor accounts for S1's greater skill in solving these kinematics problems. S2 takes each step in her solutions more slowly than S1, frequently expressing lack of confidence that she is on the right track. She takes a considerable amount of time summarizing and recapitulating the information she obtains. When she evokes a formula, she does not always substitute into it immediately the values of the variables given in the problem statement. She is less skilled and sure than S1 in both algebraic and arithmetic manipulation, and makes more arithmetic mistakes. Because of her lack of confidence, she sometimes abandons a solution attempt when she is on the right path.

Differences of these kinds might well account for the full difference in skill between the two subjects, but we have adduced evidence that another factor, too, is involved. If S1's approach to these problems may be characterized as "physical," S2's is "algebraic." There is evidence, though less decisive than we should like, that S1 generally moves from the problem statement to a representation of the physical situation, and from that representation to a set of equations. Most of our evidence for this claim is indirect -- principally the fact that S1's solution paths lie close to simple physical representations of the phenomena.

When we say that S2's techniques are "algebraic," we mean that she appears to go rather directly, in the manner of the production system with which we simulated her behavior, from the problem statements to the equations required to solve them. By studying the textbook chapter and the illustrative problems, S2 learned the algebra of kinematics, including the necessary equations for solving kinematics problems, but was only beginning to learn the physics -- how to represent complex kinematics situations. The textbook seems to have been more successful in teaching equations than in inducing a high level of physical intuition.

Conclusion

In this study we have undertaken a detailed analysis of the task of solving simple kinematics problems. We have sought to describe not only the explicit knowledge of physical laws that the student must acquire, but also how those laws must be organized and "indexed" in memory in order to provide a basis for problem-solving skills in this domain.

The physics content of the problems, which test about one week's work in a standard high school or college physics course, is quite limited, amounting to only about three or four laws and a few consequences that are derivable rather directly from them. Yet physics is not usually regarded as an easy school subject. What does the student need to learn besides the bare laws themselves? One approach to answering this question is to compare and contrast highly skilled performance on the problems with the performance of someone who is just beginning the study of physics.

Using this strategy, we have examined the protocols of two subjects working physics problems under thinking-aloud instructions. The two subjects, one an expert and one a novice, spanned a wide range of skill in both physics and algebra. We have constructed production systems that provide a first-approximation theory of the processes the subjects were using in solving these problems, and we have studied in detail both the differences between the processes of the two subjects and the deviations of each from this first-approximation theory.

The production systems that describe the behavior of the two subjects are quite similar in basic structure. The condition sides of the productions test whether the values of the independent and dependent variables in each of the physical laws are known, and trigger the action of solving the corresponding equations for the dependent variables when the appropriate conditions are met. The conditions induce a "working forward" strategy in the expert, and a "working backward" strategy in the novice.

Much of the difference in skill between the two subjects can be explained in terms of a generalized "practice effect." The skilled subject has had vastly more experience in the kinds of algebraic and arithmetic manipulations required for solving problems of these kinds. This difference in experience shows up both as difference in a variety of skills and difference in confidence.

We believe that we have also identified a more important difference that may be labeled "physical intuition." To assert that an advantage in physical intuition accounts for the superior ability of physicists to solve physics problems should occasion no surprise. Physicists and teachers of physics have been saying that for years. What we hope to have contributed in this study is a reasonably operational definition of what constitutes physical intuition, and an indication of how it enters into the solution of physics problems.

Some clear research tasks lie ahead. We must find more reliable means of detecting the presence or absence of physical intuition in problem solving behavior. We suspect that skilled subjects will provide fuller and more revealing protocols if we give them harder problems than the ones used here, and that is a direction in which we intend to move.

As a clearer picture of the nature of physical intuition emerges, it will become feasible to address some pedagogical issues. What kinds of experiences encourage the growth of physical intuition? How easy is it to learn to take the step from a physical representation of a problem to equations for solving it, and what training will facilitate that step? Is high skill in a subject like physics attainable purely with algebraic skills, and without cultivating physical intuition? How can we diagnose a deficiency in physical intuition when a high level of algebraic skill is present? These questions seem to us central ones in facilitating development and instruction in school subjects that are concerned with understanding the physical world.

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Figure 1

Protocol of S1 on Problem 19

1. A bullet leaves the muzzle of a gun at a speed of 400 meters per second.
2. The length of the gun barrel is half a meter.
3. Assuming that the bullet is uniformly accelerated
4. what is the average speed within the barrel?
5. Well, obviously one half of 400 is 200 meters per second.
6. Ah...How long was the bullet in the gun after it was fired?
7. If the average speed was 200 meters per second,
8. and the barrel is a half of a meter,
9. then it would be 100...one...wait a while..
10. The average velocity is 200 meters per second,
11. and the length is half a meter....
12. Yeh, then...ah...it's a half meter.
13. and it's 200 meters per second,
14. then it would have to be one four-hundredth of a second.

Figure 2

Protocol of S2 on Problem 19

1. A bullet leaves the muzzle of a gun at a speed of 40 meters per second.
2. V-zero equals 4C, no 400 meters per second.
3. The length of the gun barrel is .5 meters.
4. Assuming that the bullet was uniformly accelerated,
5. what is its average speed inside the barrel?
6. Its average speed inside the barrel was from zero plus...to 400....
7. Um....average velocity times the time it was there,
8. time divided by two.
9. Its average velocity was 200 meters per second.
10. Assuming that the bullet was uniformly accelerated, what is the average speed inside the barrel?
11. Its average speed was 200 meters per second.
12. That's got to be right.
13. The average speed is speed, is the speed, is the v-zero, which was zero,
14. plus v, which was the other, divided by 2.....
15. Why divided by two?
16. The average of the two speeds, right?
17. Or 200 meters per second.
18. There's something wrong with that; maybe that's what I did something wrong with on the other one. Maybe I should have taken a square or something. I'll go back and look at it in a minute.
19. How long was the bullet in the gun after it was fired?
20. How long was the bullet in the gun?
21. All right. The time.
22. If s equals v t,
23. t equals the distance divided by the average speed,
24. equals 200 meters.....
25. No, point 5 divided by 200 meters.
26. So let's worry about that.
27. Point 5 by 200 is 002.5
28. Time equals .002.25 seconds.
29. Clearly, something is wrong with Problem 18. Let's go back and do that again based on one of the sample problems.

SUBJECT S1

Prod.	Ind.Vars.	Dep.Var.	Equation
P1	vv_0T	a	$a = (v-v_0)/T$
P2	v^*T	S	$S = v^*T$
P3	Sv^*	T	$T = S/v^*$
P4	vv_0a	T	$T = (v-v_0)/a$
P5	v_0v	v^*	$v^* = (v_0+v)/2$
P6	v_0aT	v	$v = v_0+aT$
P7	ST	v^*	$v^* = S/T$
P8	v_0aS	T	$T = (-v_0+(v_0^2+2aS)^{1/2})/a$
P9	Sv_0T	a	$a = 2(S-v_0T)/T^2$
P10	v_0aT	S	$S = v_0T+.5aT^2$

Figure 3. PRODUCTION SYSTEM FOR S1

SUBJECT S2

Prod.	Ind.Vars.	Dep.Var.	Equation
P1	Sv^*	T	$T = S/v^*$
P2	v_0aS	T	$T = (-v_0+(v_0^2+2aS)^{1/2})/a$
P3	aSv_0	v	$v = (v_0^2+2aS)^{1/2}$
P4	v_0aT	v	$v = v_0+aT$
P5	v^*T	S	$S = v^*T$
P6	v_0aT	S	$S = v_0T+.5aT^2$
P7	vv_0T	a	$a = (v-v_0)/T$
P8	vv_0S	a	$a = (v^2-v_0^2)/2S$
P9	vv_0a	T	$T = (v-v_0)/a$
P10	v_0v	v^*	$v^* = (v_0+v)/2$
P11	Sv_0T	a	$a = 2(S-v_0T)/T^2$
P12	ST	v^*	$v^* = S/T$

Figure 4. PRODUCTION SYSTEM FOR S2

Table 1

Number of Words, Time and Rate Used

in Solving 19 Physics Problems

Problem Number	S1			S2		
	Number of Words	Time in Seconds	Words per Minute	Number of Words	Time in Seconds	Words per Minute
3	54	16	203	100	85	70
5	120	54	133	270	233	70
6	102	36	170	167	163	60
7	100	36	167	176	169	62
8	60	19	189	208	184	68
9	100	43	140	285	240	71
10	82	28	176	a	a	a
11	116	52	134	700[b]	513[b]	82[b]
12	50	26	115	63	48	79
13	149	79	113	591	560	63
16	351[c]	152[c]	139[c]	341	246	83
17	106	50	127	597	545	66
18	116	40	174	a	a	a
19	142	45	189	271	195	83
20	141	55	153	312	260	72
21	197	80	148	138	120	69
23	67	25	160	311	295	63
24	258	110	141	514	555	56
25	87	35	150	137	135	61

[a]Protocol incomplete.

[b]Minimum time--first part of solution lost from tape.

[c]Does not include checking time, which took 107 seconds, 212 words.

Table 2

Comparison of Solution Paths for Subjects and Simulations by Problem Type

Problem Type		Prob. #	S1		S2	
Given	Find		Protocol	Simulation	Protocol	Simulation
V_0AT	VS	5	V4-S7	V4-V*5-S1	V4-S7	V4-S7
		8	V4-V*5-S1	V4-V*5-S1	V4-S7	V4-S7
		9	V4-V*5-S1	V4-V*5-S1	V4-S7	V4-S7
		10	V4-V*5-S1	V4-V*5-S1	V4-S7	V4-S7
V_0AT	SV	11	V4-S7	V4-V*5-S1	V4-S7	V4-S7
V_0AT	S	12	S7	V4-V*5-S1	S7	S7
		20	V4-V*5-S1	V4-V*5-S1	S7	S7
		21	V4-V*5-S1	V4-V*5-S1	S7	S7
V_0VT	S	23	V*5-S1	A4-V*5-S1	A4-S7	A4-S7
		25	V*5-S1	A4-V*5-S1	V*5-S1	A4-S7
V_0VT	AS	7	A4-V*5-S1	A4-V*5-S1	A4-S7	A4-S7
		17	A4-V*5-S1	A4-V*5-S1	A4-S7	A4-S7
V_0VA	TS	6	T4-V*5-S1	T4-V*5-S1	T4-S7	T4-S7
		18	T4-V*5-S1	T4-V*5-S1	T4-V*5-S1	T4-S7
V_0VS	AT	24	a	V*5-T1-A4	A8-T7	A8-T7
V_0VS	V*T	19	V*5-T1	V*5-T1	V*5-T1	A8-T7-V*5
V_0AS	VT	16	T7-V4	T7-V4	V8-T7	T7-V8
V_0ST	V*V	13	A7-V4-V*5	V*1-A7-V4	A7-V8-V*5	A7-V8-V*5
ST	V*	3	V*1	V*1	V*1	V*1

Key: Letter is variable to be solved for; number is equation type used; e.g., V4 means "Equation 4 was used to solve for V."

Variables: A, acceleration; S, distance; T, time; V_0, initial velocity; V, terminal velocity; V*, average velocity.

Equation Type	Equivalent equations, pp. 3-4	
1	1,2	$S = V^*T$, $S = VT$ (constant speed)
4	3,4,9	$A = (V-V_0)/T$, $V = V_0+AT$, $V = AT$ (where $V_0 = 0$)
5	5	$V^* = (V_0+V)/2$
7	7,10	$S = V_0T+1/2(AT^2)$, $S = 1/2(AT)^2$ (where $V_0 = 0$)
8	8,11	$V^2-V_0^2 = 2AS$, $V^2 = 2AS$ (where $V_0 = 0$)

[a] Anomolous solution. See Text.