The Role of Problem Representation in Physics

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Since people are good at predicting the outcome of physical interactions in the world around them, why are they so bad at physics, even the branch of physics (mechanics) that deals with the interaction of everyday objects? I argue here that the process of internally simulating events so as to predict their outcome, a facility possessed by most people for common contexts, is extended and refined in a skilled scientist to become a sharp and crucial intuition that can be used in solving difficult, complex or extraordinary problems. Novices, lacking this extended intuition, find such problems difficult.

1. Problem Representations

1.1. Envisionment

The process of predicting the motion of objects in a familiar (although idealized) domain has been studied and formalized by de Kleer (1975, 1979) . He calls the process "envisionment", and it can be well illustrated through a discussion of the following problem (de Kleer, 1975).

> A small block slides from rest along the indicated frictionless surface Figure 1. Will the block reach the point marked X?

De Kleer presents the following hypothetical "protocol" analyzing the problem which corresponds to an envisionment constructed by his simulation program, NEWTON.

> The block will start to slide down the curved surface without falling off or changing direction. After reaching the bottom it starts going up. It still will not fall off but it may start sliding back. If the block ever reaches the straight section it will not fall off there, but it may change the direction of its movement.

As shown in Figure 2 the envisionment of a problem consists of a tree, with a top node corresponding to where the particle starts, with subsequent nodes produced by operators that predict the immediately subsequent motion of a particle from its current motion, and branches produced wherever two subsequent motions are possible (e.g., the block might either continue along its path, or it might slide back down).

The central features of envisionment as discussed by de Kleer are the following:

1. The entities in the problem representation are objects familiar in everyday life and appearing directly in the problem. In Figure 2 the entities include falling and sliding; envisionments for problems involving explicit interaction as well as motion would involve familiar objects (sleds, hills) and familiar interactions (e.g., touching, near).

2. The envisionment tree is developed by unidirectional operators that develop new information consistent with the passing of time. This is why the envisionment is a tree rather than a graph; there is no possibility of upward branching. This feature also implies that for each piece of new information there's only one source; there is no possibility for a single piece of information to be obtained redundantly from two other nodes.

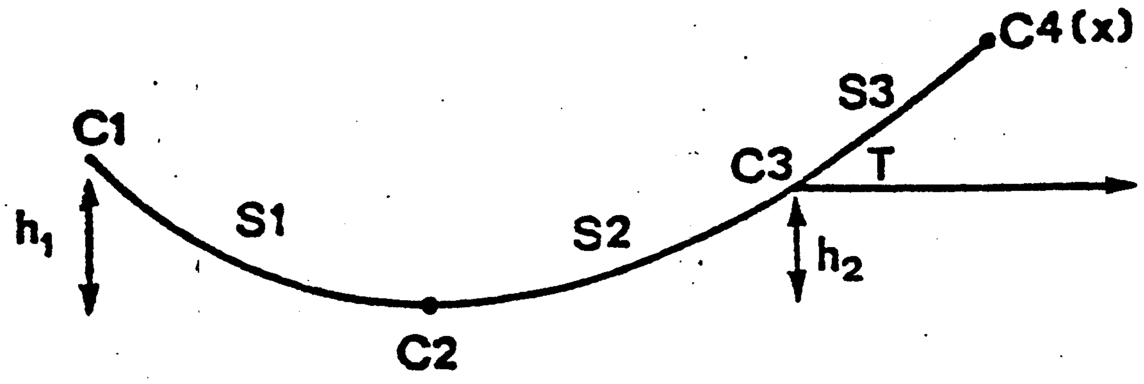

Figure 1: A particle sliding on a smooth curved path (de Kleer, 1975).

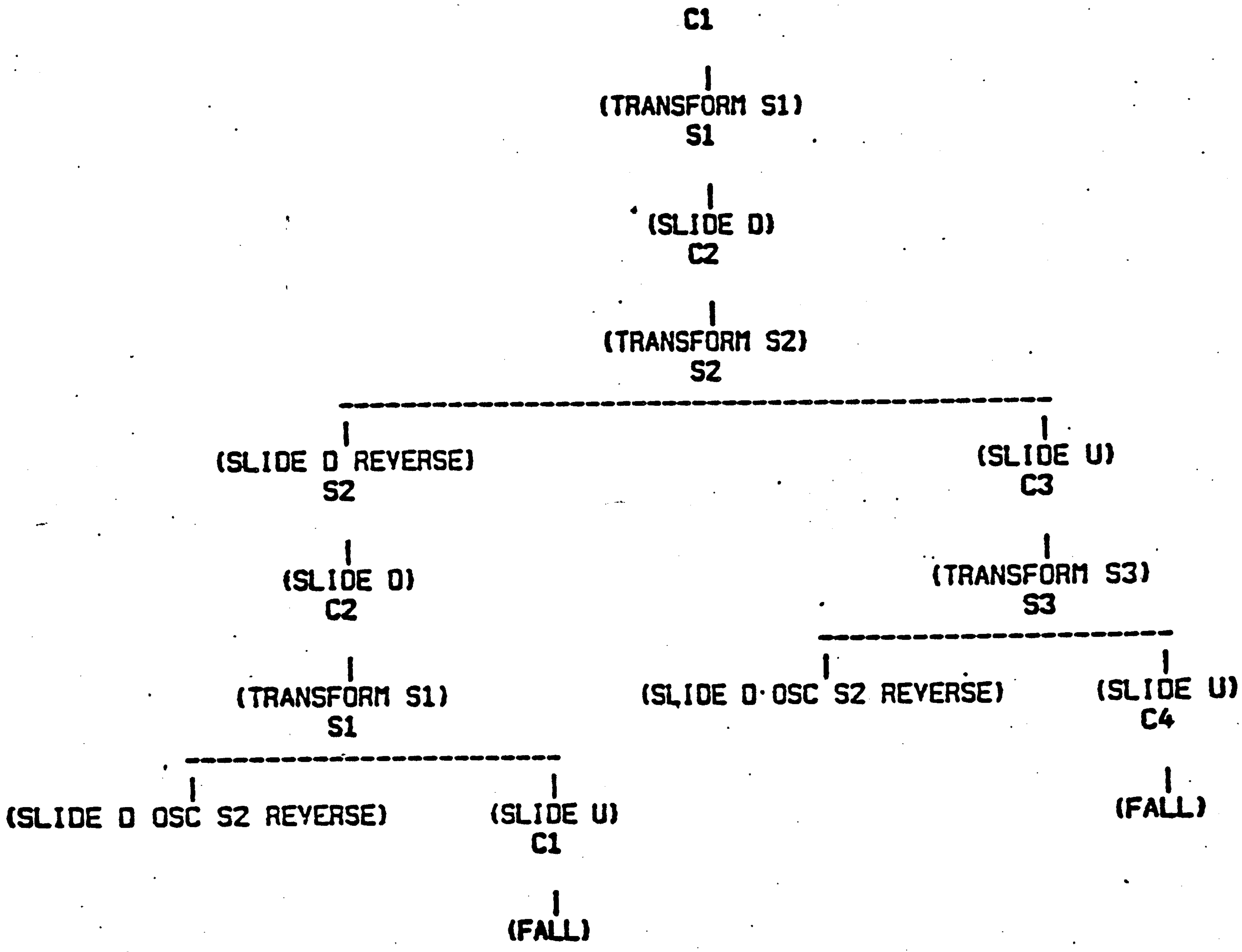

Figure 2: Envisionment given by de Kleer for the problem of the sliding block.

As de Kleer points out, envisionments would seem to be central to some aspects of problem-solving. Primarily they enable a solver to answer directly simple questions such as whether the block in Figure 1 will reach the bottom. In the NEWTON system, the envisionment also guides the process of quantitative problem solving. For example, in the sliding problem, because the initial speed on segment S3 is unknown, NEWTON determines that conservation of energy (rather than kinematics) will be used to decide whether the particle reaches C4.

For clarity in what follows I shall substitute the term *naive representation* for envisionment. Specifically a naive representation is an internal representation of the problem that contains direct representations of the familiar, visible entities mentioned in the problem, and that simulates the interaction of these entities through operators that predict subsequent events on the basis of former events, following the usual direction of time flow. This naive representation contrasts with what I shall call a *physical representation*. As discussed in the following pages, a physical representation involves entities (e.g., forces, momenta) that are not familiar but have meaning in the context of physics. This representation further has a time independent structure. Both naive and physical representations are distinct from a *mathematical* representation, a set of equations reflecting physical principles applied to the problem.

1.2. Physical Representation

I find somewhat unsatisfactory NEWTON's procedure of use of the naive representation (envisionment) to guide directly the production of mathematical equations. I think a skilled individual solving, for example, the sliding-block problem would instead create a new problem representation that involves physics entities (in this case energies), and that can be used both to make more extensive qualitative inferences and to guide more directly the mathematical solution. The following hypothetical expert protocol corresponding to Table 1, illustrates what I mean:

> The energy at C1 consists of kinetic energy, zero because the particle is at rest, and potential energy determined by the known height h_1. At C2 the potential energy is zero, because the block is at the bottom; the kinetic energy is unknown because the speed is unknown. At C3 the potential energy is determined by the known height h_2; the kinetic energy is still unknown. At some point C5 (which may be above or below C4), the particle stops and the kinetic energy is again zero. At that point the potential energy is related to the final height, which is what we want to know (because it will tell us whether the particle stops before or after C4). Thus the basic equation we want is
>
> energy at C1 = energy at C5
>
> or
>
> $0 + mgh_1 = 0 + mgh_2 + mg\ S3/\sin T.$

In Table 1, as in subsequent tables summarizing physical representations, the top line indicates the entities in the representations (here kinetic and potential energies) and the bottom line has mathematical expressions for these entities in terms of known and desired quantities. I do not think that expert solvers immediately make use of these expressions, although they are ultimately used in constructing a mathematical representation. Their presence is intended to specify how entities in the physical representation are related to known and desired quantities.

Table 1: Physical representation (in terms of energies) for the sliding block problem.

Energies:

At C1		At C3		At C5	
Kinetic	Potential	Kinetic	Potential	Kinetic	Potential
0	mgh_1	unknown	mgh_2	0	mgh_2 + S3/sin T

Table 2 compares the main features of naive and physical problem representations. Both are built by taking information from the problem and using rules of inference to create new information. Both have a strong visual aspect. In the case of the naive representation the visual aspect is essential because this representation, corresponding to de Kleer's envisionment, is an internal simulation of the events in the problem situation. It is less clear that the physical representation must always be visual, but it is worthy of comment that most physical representations seem to have this feature. Even very abstract physical phenomena (e.g., energy states of an atom, conservation of quantum properties in the interaction of elementary particles) have corresponding visible representations (energy levels, Feynman diagrams) used in solving related problems. In both naive and physical representations the inferencing rules are qualitative, and not tied directly to equations. The differences in the representations are in the kind of entities involved (familiar objects or abstractly defined physics entities) and in the rules of inference used to generate new information (operators developing new information in the timed sequence of a simulation, or operators producing new information in any order in a time independent representation).

The physical representation (unlike the naive representation) does not explicitly include time. For example, the representation in Table 1 would be the same whether the particle moved right to left or left to right or oscillated between C1 and the stopping point C5. The slots in the representation can be filled in any order either quantitatively (with the algebraic expressions in Table 1) or qualitatively by being marked as *known* (determined by known quantities) *desired* (knowledge of this quantity would provide information about the desired quantity), or *unknown* (not determined by any combination of known or desired quantities).

The physical representation of a problem is closely tied to the instantiation of the quantitative physics principles describing the problem. The energy representation in Table 1 is closely tied to the principle that the change in energy of a system is equal to the work done on it.. This close relationship

Table 2: Comparison of naive and physical representations.

Naive Representations (Envisionments)	Physical Representations
Problem representations with qualitative inferencing rules. Strong visual component.	
"Familiar" entities	Physical entities
Simulation inferencing (follows time flow)	Constraint inferencing
Distant from physics principles	Closely tied to physics principles
Tree structure, single inference sources	Graph structure, redundant inference sources
Diffused properties of entities	Localized properties of entities

makes the physical representation a means of planning a final quantitative solution because if all the slots in a physical representation can be filled, then all the corresponding parts of an equation can be written. Thus this physical representation extends the power of what de Kleer has called envisionment. Viewed in this way, the time independent nature of the physical representation corresponds to the fact that physical principles are constraint relations that can be used to make inferences in any direction, independent of time flow.

Finally, the entities in physical representation have *localized* attributes, that is one doesn't learn any more about the entity by considering the context in which it appears. This is not the case for entities in naive representations. For example, an attribute of a toboggan in a naive representation might well be that it goes down hills. However, if it does not go down a hill, the fault may be that something is wrong with the toboggan (it's improperly waxed, or full of splinters); but the fault may also be outside the toboggan (the snow is wet). Thus an attribute of the toboggan (it goes down hills) can be violated by changes not in the toboggan, but in its environment. I think this is never the case with physical entities. Nothing in the environment of a force can change any of the attributes of the force. This property of physical representations, that attributes of entities are localized to that entity and not diffused throughout the environment, is I think identical to the principle of "no function in structure" introduced by de Kleer and Brown (deKleer & Brown, 1981) as a disideratum for models of complex systems.

To illustrate the use of a physical representation, consider the following problem.

> What constant horizontal force F must be applied to the large cart in Figure 3 (of mass M) so that the smaller carts (masses m_1 and m_2) do not move relative to the large cart? Neglect friction.

The following excerpt from a typical novice protocol for this problem illustrates the difficulty

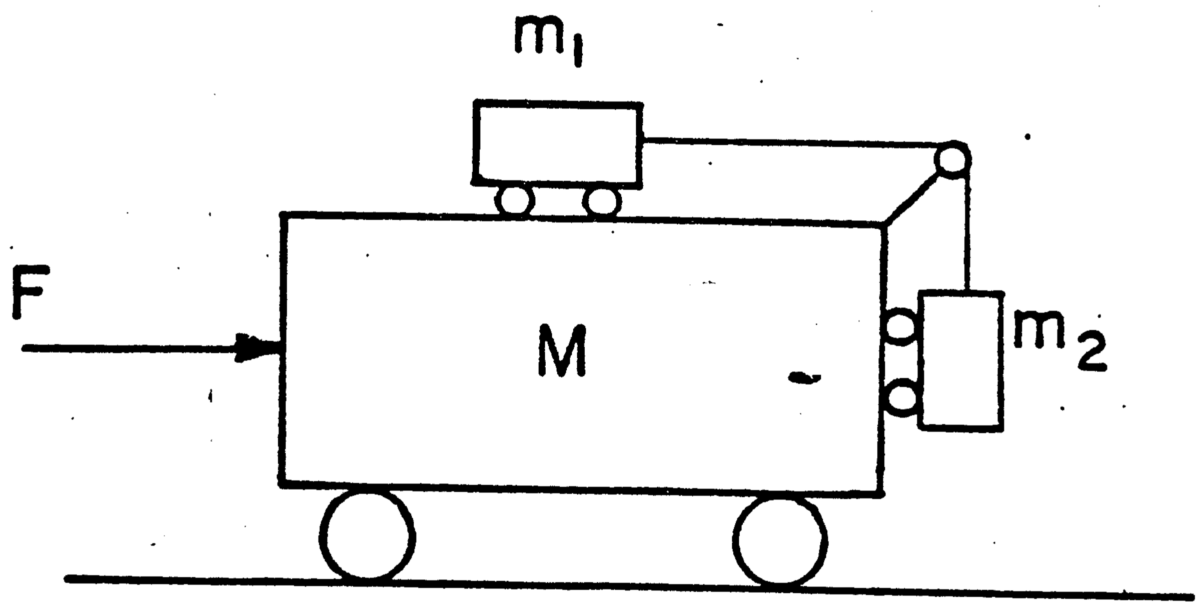

Figure 3: Three carts.

untrained individuals have in constructing even a naive representation for this problem.

> Well, I'm right now trying to reason why it isn't going to move.
> I mean I can see, if you accelerated it at a certain speed,
> the wind would push on m_1 so m_2 wouldn't fall.
> (later)
> Once I visualize it, I can probably get started.
> But I don't see how this is going to work.

This subject, like most other novices, never did succeed either in understanding what was going on in the problem or in solving it. In contrast, consider the following excerpt from the protocol of an expert subject solving the same problem using a physical representation involving reference frames and pseudo-forces.

> Well, with a uniformly accelerating reference frame, all right?
> So that there is a pseudo-force on m_1 to the left
> That is just equivalent --
> Just necessary to balance out the weight of m_2.

This subject proceeded immediately to write an equation equating the pseudo-force on m_1 and the tension force due to the weight of m_2 and then to solve the problem. This subject and all the other expert subjects looked at this problem not as a confusing collection of carts and ropes and pulleys, but as an abstracted object (of mass m_1) at rest at an accelerated reference frame (of the large cart), and therefore acted on by a pseudo-force to the left as well as the obvious tension force to the right. [The so-called pseudo-force in physics is that "force" that you feel snapping your head back in the accelerated reference frame of a car starting quickly from a stoplight. It is not a true force due to another object, but a fictitious force experienced because the frame of reference (e.g., the car) is accelerating.] The fact that the object is at rest means that these two forces must be equal in magnitude. Thus the *physical* representation used by the skilled subjects to solve this problem seems profoundly different from *naive* representation used by the novice subjects or the envisionments used by NEWTON.

One of the most powerful features of the physical representation is the redundancy of the inferencing rules. For example, Table 6a shows one physical representation for the cart problem. The existence of the tension force and the fact that the acceleration of the small cart is zero (relative to the large cart) implies that there is some force directed towards the left. This force is also implied by the acceleration of the reference frame to the right, and therefore the existence of a leftward pseudo-force $\mathcal{F}$. Thus a solver using such an inferencing scheme has a double chance to find this force. If the pseudo-force rule was forgotten, the existence of this force might still be inferred (and the pseudo-force rule remembered) due to the redundant inferencing rule.

1.3. Schemas for Producing Physical Representations

How are physical problem representations, like those discussed above, constructed? The following paragraphs give examples of two plausible schemas for constructing physical representations in mechanics. For each the inferencing rules have been divided into *construction* rules that act on the original (naive) problem representation to product entities in the physical representation, and *extension* rules that act on an existing physical representation to add new entities.

Forces Schema

This schema corresponds to the physical principle that the total force on a system (along a particular direction) is equal to the system's mass times its acceleration (along that direction). The schema applies to the following situations.

1. A system that has zero acceleration (along a particular direction) and is acted on by two forces (parallel to that direction) that have equal magnitudes and opposite directions.

2. A system that has a non-zero acceleration (in a particular direction) and is acted on either by one force in that direction, or by two opposite forces, the one with larger magnitude being along the acceleration.

Construction Rules. These rules correspond to force laws that allow inference of the direction and magnitude of a force from the characteristics of the object exerting it. For example, on an object of mass m, a force law states that the earth exerts a downward force of magnitude mg, where g is a known constant. The remainder of the construction rules correspond to what are called *kinematic* principles, that relate the acceleration of a system to other descriptions (e.g., velocity, distance travelled) of its motion.

Extension Rules. There are two complementary rules: If the acceleration is zero, and one force is known, then there is another force of equal magnitude and opposite direction. If there are two forces of equal magnitude and opposite direction, then the acceleration is zero. [For simplicity these rules are stated in terms of just two forces. All can be generalized by letting each force equal the sum of all the forces in that direction.]

Work-Energy Schema

This schema corresponds to the physical principle that the total energy of a system changes by an amount equal to the work done on it.

Construction Rules. These rules correspond to principles describing the various kinds of energy a system (or component of a system may have). For example, the kinetic energy of a particle is determined by its mass m and speed v, and is quantitatively equal to $\frac{1}{2}mv^2$. An additional rule relates the work done on a system to the "non-conservative" forces on the system and to the path it travels.

Extension Rules. Consider the initial energy of a system, its final energy, and the work done on it in the relevant interval. If any two of these are known, then the third can be found.

Example

As illustration of the application of these schemas, consider the problems stated in Table 3, and the corresponding physical representations summarized in Table 4. Problem 1 concerns a block. In a physical representation involving forces, the block is acted on by two forces having magnitudes $F_1 = mg \sin\theta$ and $F_2 = \mu mg \cos\theta$. It has an acceleration which can be shown to be $v^2/2l$. Using energies, the physical representation of the same problem involves an initial energy $E_1 = mgl \sin\theta$ of the block at the top of the ramp, a final energy $E_2 = \frac{1}{2}mv^2$ of the block at the bottom of the ramp, and a work $W = \mu mg \cos\theta\, l$ done on the block during the intervening time. The physical representations of problem 4 are analogous.

2. Empirical Studies

The remainder of this paper discusses empirical studies relevant to how physical representations function in problem solving and how expert-novice differences in problem-solving performance can be explained by saying that experts are much better able to construct and use physical representations.

2.1. The Order of Principles Applied in Easy Problems

In solutions to very easy problems, the use of a physical representation is mainly visible in the order in which principles are applied. Consider the two problems in Table 3. Physical representations can be constructed for these problems using either forces or energies.

Table 3: Two easy problems.

1. A body of mass m starts from rest down a plane of length l inclined at an angle θ with the horizontal. If the coefficient of friction is μ, what is the body's speed as it reaches the bottom of the plane?

4. What is the minimum stopping distance for a car traveling along a flat horizontal road, with initial speed v_0, if the coefficient of friction between tire and road is μ?

Table 4 shows physical representations for these problems constructed using either the force or work-energy schemas. Using the force schema, the slots are the acceleration and the constant force(s) exerted on the objects of interest (in problem 1 a component of the gravitational force and a frictional force, in problem 4 just a frictional force). Underneath the label for each slot is the value filling this slot in that problem (e.g., mg sin θ for the gravitational force component in problem 1.) In the representation constructed using the work-energy schema, the slots are an initial energy, a final energy, and an intervening work.

Eleven expert and eleven novice subjects completed solutions to these problems while thinking aloud to provide a description of their work. The novice subjects were students at the University of California, Berkeley who had completed about eight weeks of their first university-level physics course (an introductory physics course for students in physics and engineering). The expert subjects were professors and advanced graduate students in physics at the University of California, Berkeley, who had taught an elementary mechanics course within two years.

For the two problems in Table 3 seven of the novices and ten of the experts provided solutions that could be interpreted. (Two other papers (Larkin, McDermott, Simon, & Simon, 1980, Larkin, 1981) provide details of this study.)

If the expert subjects are using the schemas summarized in Table 4, one would expect the expressions corresponding to slot entries to appear as intermediate results in the problem solution. Thus one would expect in a force solution to problem 1 some statement that

$$mg \sin \theta$$

is the relevant component of the gravitational force. In contrast a solution violating this expectation might include

$$F_g = mg$$

connected to other information by

$$F_g/\sin \theta = ma + f.$$

In Table 4 a number (1,2, or 3) indicates that the subject referenced by the number at the left used as an intermediate result the schema-slot entry listed above. The value of the number is the order in which these results were stated. Clearly intermediate results generally do correspond to schema-slot entries. Also known schema slots are generally filled before those related to desired quantities. Thus the result reported elsewhere (Simon & Simon, 1978, Larkin, McDermott, Simon, & Simon, 1980, Larkin, 1981) that experts tend to work forward, to "develop knowledge" is reinterpreted here by saying that experts fill slots in a schema to make a physical representation, starting with slots related to known quantities.

In contrast, novice solutions to the same two problems do not show an order corresponding to filling slots in a force or energy schema. Instead their solutions are consistent with the hypothesis that novices solve the problem using not a physical representation but a mathematical representation involving equations representing the physics principle. In this representation, as reported extensively elsewhere (Simon & Simon, 1978, Larkin, McDermott, Simon, & Simon, 1980), novices seem to use a

Table 4: Schema slots and fillers for problems in Table 3. Numbers indicate the order expert subjects filled the slots.

(a) Force Representation

	Problem 1			Problem 4	
	F_1	F_2	a	F_1	a
	mg sinθ	μmg cos θ	$v^2/2l$	μmg	$v^2/2xl$
S1:	1	2	3	1	2
S2:	2	1	3		
S3:	1	2	3	1	2
S5:	2	1	3	1	2
S6:	1	2	3		
S7:	1	2	3		
S8:	1	2	3	1	2
S11:	1	2	3	1	2

(b) Work-Energy Representation

	Problem 1			Problem 4		
	E_1	E_2	W	E_1	E_2	W
	mg$l$ sin θ	$\frac{1}{2}mv^2$	μmg cos θ l	$\frac{1}{2}mv^2$	0	$\mu g l$
S2:				2	-	1
S4:	3	2	1	2	-	1
S6:				-	1	2
S7:				-	1	2
S9:	2	3	1	1	-	2
S10:	2	3	1	1	-	2

means-ends strategy involving writing an equation, assessing differences between this equation and an equation that would provide the desired answer, and then taking steps to eliminate this difference. Specifically, novices first write an equation involving the desired quantity (speed v in problem 1, distance x in problem 4). They then assess the resulting equation for any quantities which are unknown and therefore can not appear in the final expression. This cues access of a new equation that contains this undesired quantity and therefore allows substitution for it.

Figure 4 shows principles connected by lines indicating the order in which this means-ends strategy selects these principles for application. The structure is a tree, with an equation cueing the access of one equation (or more than one alternative equations) each containing a quantity that was unknown in the preceding equation. The numbers underneath each branch of the tree describe the order of principles selected by the novice subjects. In general, their work can be accounted for quite well by the algebraic means-ends strategy reflected by the tree.

2.2. Combination of Schemas in Harder Problems

What happens when experts and novice subjects solve harder problems, for example, those in Table 5?

The same subjects who solved the easier problems (Table 3) also solved those in Table 5. A total of 11 (out of 11) novice subjects and 8 (out of 11) expert subjects produced interpretable solutions to these problems. (Only the initial sections of the novice protocols were considered.)

Table 5: Two hard problems.

3. What constant horizontal force F must be applied to the large cart (of mass M) so that the smaller carts (masses M_1 and M_2) do not move relative to the large cart? Neglect friction.

5. A particle of mass m is suspended from a frictionless pivot at the end of a string of unknown length, and is set whirling in a horizontal circular path in a plane which is a distance H below the pivot point. Find the period of revolution of the particle in its orbit.

One explanation of why these problems are difficult is that their physical representations require coordination of more than one of the schemas described earlier. Considering first problem 3, Table 6 shows the two alternate physical representations used by the expert subjects to solve this problem. Part (a) shows a physical representation constructed by applying the force schema to the following two aspects of the cart problem. Schema 1 involves cart 1 which is at rest (acceleration a_1 zero) relative to the accelerated reference frame of the large cart. This is because two equal and opposite forces (the tension, equal to m_2g, and the pseudo-force $\mathcal{F}$ due to the acceleration a of the reference frame) are balanced. Schema 2 focuses on the system of all three carts, and relates the total force on this system (just the applied force F) to the acceleration a and the total mass

Problem 1: Desired quantity v

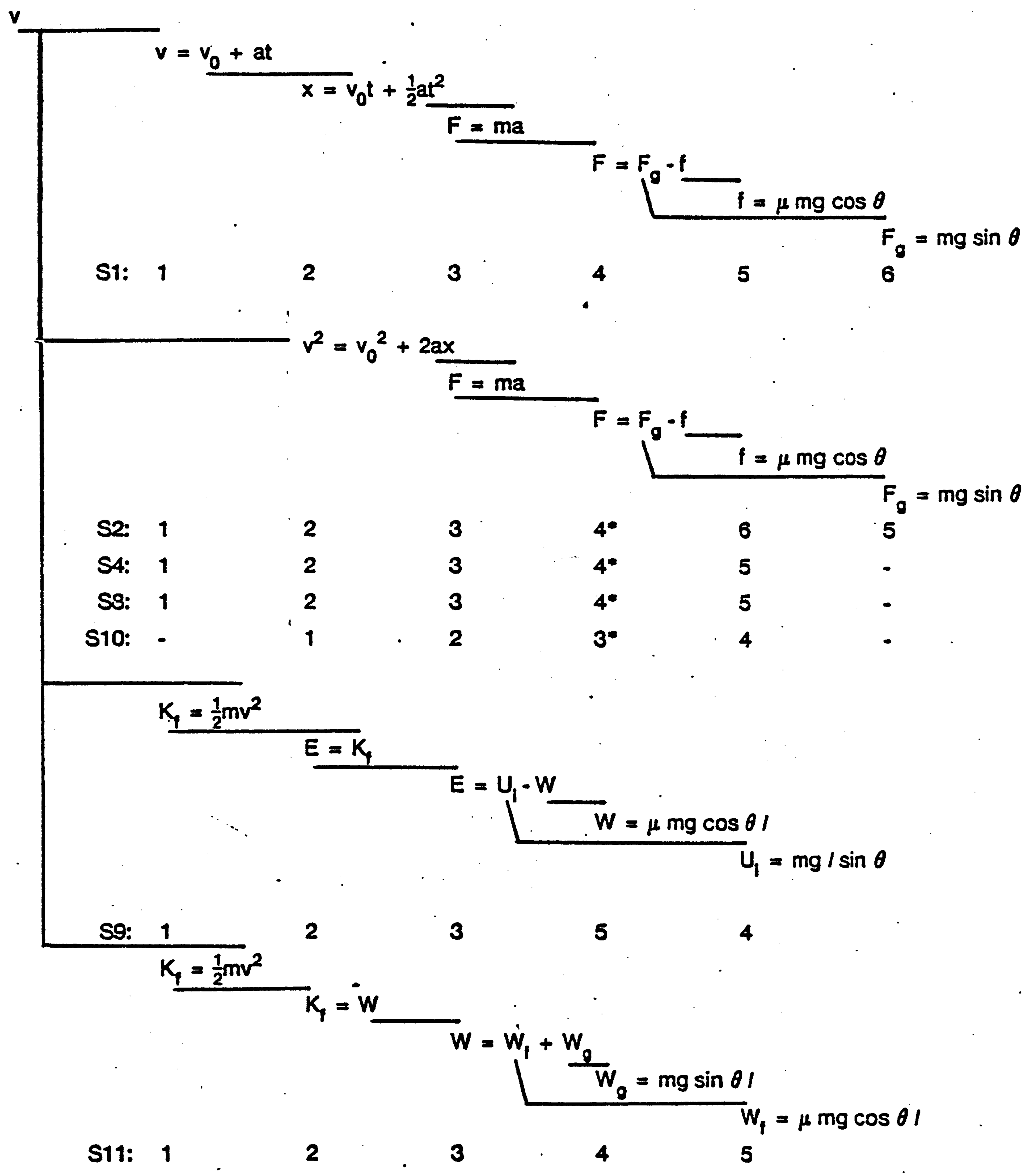

Figure 4: Tree of relations generated by a means-ends strategy for problems in Table 3. Numbers indicate the order in which novice subjects accessed these equations.

Problem 4: Desired quantity x

x

$x = v_0 + \frac{1}{2}at^2$ — $v = v_0 + at$ — $F = ma$ — $F = f$ — $f = \mu mg$

	$x = v_0 + \frac{1}{2}at^2$	$v = v_0 + at$	$F = ma$	$F = f$	$f = \mu mg$
S1:	2	3	4	5	1

$v^2 = v_0^2 + 2ax$ — $F = ma$ — $F = f$ — $f = \mu mg$

		$v^2 = v_0^2 + 2ax$	$F = ma$	$F = f$	$f = \mu mg$
S2:	3	2	4	5	1
S4:	2	3+	4	5	1

$W = Fx$ — $F = \mu mg$ — $W = \frac{1}{2}mv^2$

	$W = Fx$	$F = \mu mg$	$W = \frac{1}{2}mv^2$
S8:	2	1	3
S9:	1	2	3
S10:	3	1	2
S11:	2	1	3

* Used incorrect version of principle (see Larkin, 1981).

+ Accessed both $v^2 = v_0^2 + 2ax$ and $v = v_0 + at$.

(Figure 4: continued)

$\mathcal{M}$ = M + m_1 + m_2 of this system. Thus the total horizontal applied force F is equal to the total mass of the system times acceleration *a*. Table 6b shows an alternate physical representation for the problem in which cart 1 is viewed as having an acceleration *a* relative to the inertial reference frame of the ground, an acceleration caused by the tension force T which is equal to m_2g. The other part of this physical representation is identical to that in part (a).

As in Table 4, Table 6 shows the slots and instantiations associated with each of these schemas, and the order in which these schema elements were mentioned by each of the expert subjects. As one would expect if these schemas are guiding experts' work, in all cases the work associated with one schema is completed before work associated with another is begun.

The situation for problem 5 is similar. The coordinated force schemas used to solve this problem are shown in Table 7. [T is the tension force of the string; θ is the angle between the string and the vertical; R is the radius of the circular path; τ is the period of the motion.] The two relations at the right do not come from any schema I currently recognize -- one is simply a relation among quantities describing circular motion, the other an application of a trigonometric definition. As the order-numbers below the schemas indicate, expert subjects do seem to use these schemas. Furthermore, they generally complete the related force schemas before proceding to the extra kinematic and trigonometric relations.

Turning to novice performance on the same two difficult problems, if novice solvers work with a mathematical representation, then their solutions should begin with expressions involving the desired quantities. For difficult problems, their solutions should end when they can no longer think of any relations to substitute. This result is most striking in the novice performance in problem 5. Ten out of eleven subjects began their work with a relation involving the desired quantity, τ the period of revolution. (No expert subject did so.) The just two subjects who did complete the solution did so with a completely means-ends dictated order.

In problem 3 the situation is somewhat different. The only equation containing the desired quantity is F = ma applied to the composite system of all three carts. Apparently novice subjects do not readily see this application, as none of them began with this equation.

Blocked from their usual means-ends approach, novice subjects were apparently driven to try to use a physical representation. The result is shown in Table 8. The representation is a single force schema applied to cart 1, which is at rest with acceleration zero, and is acted on by a tension force equal to m_2g. As indicated by the presence of order-numbers in Table 6, most subjects completed that much of the representation. They then knew that an additional left-directed force F_1 had to exist to keep the cart 1 at rest, and many expressed puzzlement over what it could be. However, as indicated in Table 6, most completed this final schema slot by inserting (incorrectly) the most salient

Table 6: Schema slots and the order expert subjects filled them for problem 3 in Table 5.

(a) Accelerated Reference Frame

	Schema 1			Schema 2		
Slots	F_1	F_2	a_1	F_1	a	$\mathcal{M}$
Fillers	$F = m_1 a$	$T = m_2 g$	0	F	a*	$M + m_1 + m_2$
S2	4	5	-	2	1	3
S5	2	1	-	3	5	4
S9	4	5	-	1	2	3

(b) Inertial Reference Frame

	Schema 1		Schema 2		
Slots:	F_1	a	F_1	a	$\mathcal{M}$
Fillers:	$m_2 g$	a*	F	a*	$M + m_1 + m_2$
S3	2	1	4	-	3
S4	1	2	3	5	4
S6	1	2	3	5	4
S7	1	2	4	3	5
S8	2	1	3	4	5

*These accelerations equal.

force, the desired force F.

Table 7: Schema slots and the order expert subjects filled them for problem 5 in Table 5.

	Horizontal		Vertical			Kinematics	Trigonometry
	F_1	a	F_1	F_2	a		
	T sin θ	v^2/R	T cos θ	mg	0	$v = 2\pi R/\tau$	$H = R \tan \theta$
S2:	1	2	5	6	-	3	4
S3:	3	4	1	2	-	6	5
S4:	3	4	1	2	-	5	6
S5:	3	4	1	2	-	5	6
S6:	3	4	1	2	-	5	6
S7:	3	4	1	2	-	6	5
S8:	3	5	1	2	-	6	4
S9:	2	1	3	4	-	5	6

Table 8: Schema elements used by novice solvers of problem 3 in Table 5.

	F_1	F_2	a
	F	$T = m_2g$	0
S1:	2	1	-
S2:	-	-	-
S3:	2	1	-
S4:	-	-	-
S5:	-	-	-
S6:	2	1	-
S7:	2	1	-
S8:	-	1	-
S9:	-	1	-
S10:	-	1	-
S11:	-	1	-

2.3. Schema Selection in a Very Hard Problem

The problems considered in the preceding discussion involve the use of multiple schemas to construct a physical representation, and are sufficiently difficult that most novice solvers could not complete them. However, these problems are still straightforward for expert subjects, who quite immediately selected and applied the appropriate schemas. What is the situation if we turn to even more difficult problems, problems where even the selection and application of an appropriate schemas is difficult? For example, consider the following problem:

> A loop of flexible chain, of total weight w, rests on a smooth right circular cone of base radius r and height h. The chain rests in a horizontal circle on the cone, which has a vertical axis. What is the tension in the chain?

Six expert subjects, advanced graduate students or faculty in the Physics Department of the University of California, Berkeley, attempted to solve this problem. One subject became thoroughly confused and his data was not analyzed.

The main features seen in the five analyzed protocols are illustrated by the excerpt from a sample protocol shown in Table 9. After reading the problem and constructing a naive representation for it, This subject initially tries to construct a physical representation involving forces on a small element of the chain. This attempt is based on two decisions: the decision to use forces, and the decision to focus on a small element of the chain. Apparently these decisions were made quite rapidly as they are reflected by no statements in the protocol.

Table 10 shows why the subject ultimately abandons this representation attempt. He knows the element of chain is at rest and thus expects the physical representation to show forces balanced in both directions. Although, because of the curvature of the cone, the tension force actually has a small component directed toward the left, the subject assumes that the tension force is perpendicular to the plane of the paper. Thus he can not find any force balancing the horizontal component of the force S' due to the surface of the cone. As he realizes this, there is a long pause in the protocol during which he apparently selects another possible schema for constructing a physical representation.

As indicated in the labels for the protocol in Table 9 each attempt is based on a decision about the kind of physical schema to use (forces, virtual works) and about the aspect of the problem (element of chain, whole chain, half of chain) to be considered. In all cases but the last he fails to complete the physical representation, abandons the effort, and selects an alternative type of physical representation. With the fourth attempt, in which he applies a force schema to half of the chain (Table 9), he is finally successful in achieving a physical representation in which the various inferences involving schema slots are consistent. That is the horizontal rightward force due to the cone surface is balanced by the two leftward tension forces on the ends of the chain, thus accounting for the zero acceleration of the chain (Table 10). He then proceeds immediately to write corresponding equations that lead to a correct solution.

All five protocols were analyzed by considering just those statements made before the subject wrote his first equation, presumably statements corresponding to the construction of physical representations. These statements were subdivided into sections, each corresponding to a single

Table 9: Protocol of an expert subject solving the cone problem.

Naive Representation
[reads problem]
So I draw a right angle, or a triangle, such that at the bottom is r and the side is h. And the chain is resting at some point. I'll just draw a cross section of it.

Force Representation of Small Element of Chain
The tension...um is acting perpendicular to the plain of the paper.
Let me think about that a minute. Well let me think about that a minute.
Tension is always acting in the same direction as the direction of the...
Yeah, it has to be perpendicular to the plane of the paper.
I'll think about it. I'll draw a dot there.
And then there will be another force on the chain, which will be perpendicular to the edge of the cone. Which will be something. And the upward component is something.
Let's call that something S. Will also be equal to mg.
Because, because, the chain isn't moving up or down.
So the other component is going to be...something else...something else.

[long pause]

Virtual Work Representation of Whole Chain
I'm starting now to think sort of of virtual work.
But that's not going to get me anywhere, because...
[qualitative statements omitted]
[long pause]

Force Representation of Small Arc of Chain
All right, well look.
If I look down on the cone from the top, I get a circle.
And the tension at any point there is pulling in opposite directions.
And if I look at a short arc of that...
[qualitative statements omitted]
Um...the problem is that in any infinitesimal cross section of the chain, the two tension forces almost perpendicular, not parallel.

Force Representation of Half Chain
Well, but if I cut the circle in half, I can say that
the total tension acting on one half of the chain in one direction is 2T.
So there's a force acting in that direction which is 2T.
Meanwhile the force acting in the other direction,
which is a component of S,
is going to be...I'd have to integrate.
[applies this chunk to begin generating equations.]

Table 10: Physical representations of the cone problem described by the protocol in Table 9.

Element of Chain

Vertical Forces			Horizontal Forces		
F_1	F_2	a	F_1	F_2	a
component S'	mg	0	??	component S'	0

Half Chain

Horizontal Forces		
F_1	F_2	a
2T	integrated component S'	0

attempt to build a physical representation, specifically to a decision about a type of representation (forces or works) and an aspect of the problem to be considered.

Each protocol contains one or more attempts at physical representations, with the last attempt followed by a corresponding mathematical representation that leads to a correct solution. Two different patterns were seen in the attempted physical representations. Two subjects immediately selected the one representation (virtual work applied to the whole cone) that in fact leads to the most direct solution. This selection was accompanied by a statement like "I know how to do this one. It's virtual work." After completing the slots of the physical-representation schema, these two subjects proceeded immediately to a corresponding correct mathematical solution. Apparently for these subjects this problem was *not* difficult, because they immediately knew what physical representation it involved. The remaining three subjects each considered two or more possible physical representations elaborating each very much as illustrated in Table 9. Only the last representation was accompanied by equations.

In summary then, expert subjects confronted with a difficult problem selected a physical representation through successive attempts. While the mechanism for initially selecting a schema for such a representation is not clear, it seems to be rather quick and uncritical. Then considerable effort is expended to develop or fill the slots in that representation. These slots are filled through redundant inferencing paths. For example, in the first part of the protocol in Table 9 the subject infers that there must be a force directed to the left (because the chain-element is at rest), but is unable to infer a value for that force. If such inconsistencies arise, the representation schema is abandoned and another selected and tried. Only when a schema is successfully instantiated to form a complete consistent representation is it accepted and translated into a mathematical representation.

3. Possibilities for Instruction

3.1. The Effect of Teaching Physical Representations

If the ideas discussed in the preceding sections have validity, then is should be possible to improve success in problem solving by training students directly to represent problems using physical schemas. In one experiment described here this possibility was assessed the following way:

Beginning physics students were taught to apply individually seven principles describing DC circuits. After this initial instruction, a randomly selected half of the students received additional training on physical representations while the remaining students received training in systematically applying these principles to generate equations, and in algebraically combining these equations to obtain the desired results. After training, each student attempted to solve three problems requiring the joint application of several of the individual principles he had studied. Further details of this study are given in an earlier paper (Larkin, 1977).

The ten subjects were volunteers enrolled in the second quarter of a three-quarter, calculus-based physics course for students interested in biology and medicine. They were prepared to study DC circuits, but had no exposure to this area before the experiment. The seven principles used in the experiment included the equivalents of Kirchoff's laws, principles describing resistors and emf sources ($V = RI$, $V = -\mathcal{E}$), and a few definitions. To learn the individual principles, all subjects used a programmed booklet, and worked unsupervised for about three to six hours. During the first part of each experimental session each subject demonstrated the ability to apply each principle correctly to generate an equation. If the subject made errors or seemed uncertain, additional explanation and practice were provided.

Five randomly selected subjects saw diagrams illustrating two ways of constructing physical representations of DC circuits using the analogy between electric currents and fluid flow, and the analogy between potential and height. This instruction was intended to enable the students to do some of the things experts do in thinking about simple circuits -- visualize how the current combines and separates in the wires; relate the potential changes in various parts of the circuits. The remaining five subjects spent an equivalent amount of time cataloguing principles by grouping together principles containing the same quantity; and studying an algebraic strategy for systematically relating the various currents in the problem, then the potential drops, and finally the currents to the potential drops.

Each test problem required use of several of the seven principles, sometimes to more than one aspect of the problem. While working on the problems, each subject had available a chart summarizing the seven principles. Subjects talked aloud as they worked, and errors were identified and explained without indicating what should correctly be done. A problem was considered "solved" if the subject produced a correct solution within a time limit of 15 or 20 minutes.

The number of problems solved by these two groups was strikingly and significantly different $P < 0.05$ using a small-sample version of the Mann-Whitney test. Of the subjects receiving experimental training, three solved all three problems, and two solved two problems. Of the remaining subjects (receiving the control training), four of the five subjects solved no more than one problem. (The fifth subject solved all three problems.)

3.2. Physical Representations in a Textbook

The preceding experiment directly manipulated the teaching of physical representations. To conclude the discussion on instructional possibilities, I describe briefly a preliminary study of one subject learning from an existing physics textbook. She was asked to study a section describing accelerated motion along a straight line in a widely used physics text (Halliday & Resnick, 1966) pp.38-44, and to work the problems at the end of the chapter marked as corresponding to these sections. She talked aloud as she studied and worked the problems for a total of 3 hours. Relevant to the central thrust of this paper, that the ability to construct physical representations is central to skill in problem solving, several features of her protocol, particularly in the first hour that involved reading the text, are very striking.

Her work while studying these sections was divided into the categories listed in Table 11. I searched for, but was unable to find, any statements reflecting development of qualitative inferencing, links that might be used in building the physical representation. Specifically, I looked for any qualitative interpretations of the central physical relations presented in the chapter. For example, the equation presented in the chapter as

$$x = x_0 + v_0 t + \frac{1}{2} at^2$$

can be written as

$$x - x_0 = v_0 t + \frac{1}{2} at^2$$

and interpreted by a sentence like: The distance travelled by a particle moving with constant acceleration is equal to the initial velocity times the time, plus or minus an extra term reflecting the fact that it is accelerating or deaccelerating. No such statements were found.

This subject spent an enormous amount of time trying to understand the graphical

Table 11: Times for various tasks while reading a physics text.

Reading and comprehending prose text.	16 minutes
Doing algebra.	11 minutes
Studying and comprehending graphs.	26 minutes

Table 12: Table of kinematics relations given in Halliday & Resnick (1966).

Equation	Contains x	v_x	a_x	t
$v_x = v_{0x} + at$	X	✓	✓	✓
$x = x_0 + \frac{1}{2}(v_{x0} + v_x)t$	✓	✓	X	✓
$x = x_0 + \frac{1}{2}at^2$	✓	X	✓	✓
$v_x^2 = v_{x0}^2 + 2a(x - x_0)$	✓	✓	✓	X

presentations in the text. Most of this time (about 15 minutes) was spent on the initial set of graphs describing non-uniform acceleration. These graphs are indeed very complex and the subject's protocol revealed serious misunderstandings. Furthermore, although in two of the problems making graphs are suggested, the subject was never able to use any graphical techniques to help her in solving problems.

The section of text most relevant to the problems solved, and the only section used by the subject as she solved the problems, is that presented in Table 12. The striking feature about this section is its emphasis on a mathematical equation-based representation.

In summary, it doesn't seem surprising that students who have studied a text similar to that discussed in the preceding pages tend not to construct or to use physical representations. In this case the physical representation discussed (graphs) was both confusing and not relevant to the problems. The material most relevant to the problems was in completely mathematical form. The subject herself made no interpretations of the equations that might have been used in making physical representations.

4. Summary

I have described roughly an analog to de Kleer's *envisionment* which I think begins to account for how skilled individuals reason qualitatively or intuitively about complex situations. This *physical representation* has the following features:

- The entities are technical, with meaning only in physics.
- The inferencing rules are qualitative.
- The inferencing rules are time-independent and redundant.
- The representation is closely associated with fundamental principles of physics.
- Properties of the entities are localized to those entities.

This representation was illustrated by preliminary versions of rules for creating and extending two kinds of physical representations (force and work-energy).

Empirical studies relevant to this conceptualization of expertise include the order in which experts and novices access principles in solving simple problems, and the extent and nature of the qualitative discussion preceding any quantitative work on a more difficult problem. In the easy problems, experts seemed to use principles in an order dictated by the schemas they were completing, while novices either followed an order based on a mathematical or were unable to do the problem. In more difficult cases, experts seemed to assess whether a schema could be completed without contradictions before doing any quantitative work. Once this was ascertained, then the quantitative work proceeded uneventfully.

Finally I have suggested how work is beginning to elucidate how physical representations might function in instruction. One primitive study obtained dramatic results by training subjects essentially to use physical representations. Yet study of a popular textbook, and of one subject studying that text, suggests that information relevant to such representation is sparce and hard to apply to problems.

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