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Models of Competence in Solving Physics Problems*

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We describe a set of two computer-implemented models that solve physics problems in ways characteristic of more and less competent human solvers. The main features accounting for different competences are differences in strategy for selecting physics principles, and differences in the degree of automation in the process of applying a single principle. The models provide a good account of the order in which principles are applied by human solvers working problems in kinematics and dynamics. They also are sufficiently flexible to allow easy extension to several related domains of physics problems.

1. INTRODUCTION

We investigate here the nature of competence in the domain of solving textbook physics problems at the college level. The domain is appealing for two reasons. First, like all classroom domains, it captures some features of "real-world" problem solving, but compresses these features into a context manageable in a classroom or a laboratory. Second, in comparison with other sciences, physics has a particularly taut and transparent organization around a relatively few general principles. Thus it may be possible to see in this well-structured domain, general features of skill in solving problems, which would be difficult to see in other domains.

Our approach has been to look in considerable detail at the problem-solving processes of individuals with varying amounts of experience in solving such problems. Specifically, we have collected and analyzed think-aloud protocols of individuals ranging in skill from college freshman to established physicists, and

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on problems from two related areas of physics. In addition, we have been able to account for data involving an especially difficult problem, and involving a third and rather different area of physics.

On the theoretical side, we have built two models (simulation computer programs) that solve the same problems that our subjects did. We have tried to keep these models plausibly consistent both with what is known about human information processing and with the rough qualitative features of what we see people do in solving the problems. Thus, for example, we have restricted ourselves to a reasonable size for short-term memory, and have used an external memory in a fashion consistent with what human solvers write on paper.

Finally, we have matched the data from the human subjects against the output of our models. Since one model preferentially accounts for data from less skilled subjects, and the other for data from those with more skill, the two models reflect naive and more competent problem-solving behavior.

In this paper we first describe the main features of the two models, identifying both the basic knowledge needed to solve physics problems at all, and the different knowledge in the more and less skilled versions of the model. Then we match the problem solutions produced by these models against the work of more and less skilled human solvers, and discuss how the models can be extended to different and more demanding problems with similar matches against samples of human data. Finally we speculate on how the simple features of competence captured in our models might figure in a more extensive theory of skilled problem solving.

To make more concrete the kind of tasks we are talking about, consider the following problem, one of those used in our study.

Problem 1. A block of mass m starts from rest down a plane of length l inclined at an angle θ with the horizontal. If the coefficient of friction between block and plane is μ, what is the block's speed as it reaches the bottom of the plane?

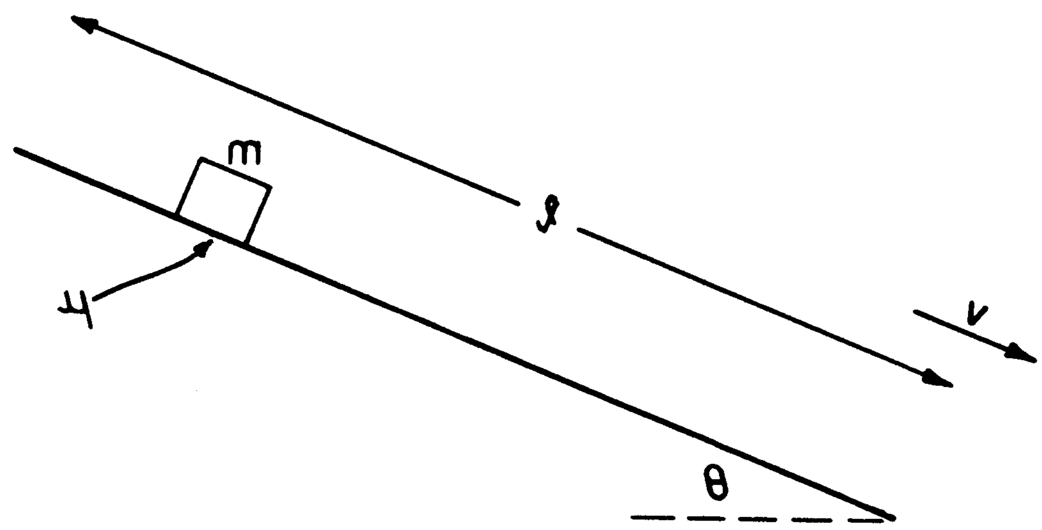

Figure 1-1: Typical sketch for a physics problem.

Figure 1-1 shows a typical sketch that might be constructed by a skilled human solver working toward a solution like the following:

> The forces on the block, acting parallel to the plane, are mg sin θ due to gravity, and μ mg cos θ due to friction, where g is the known gravitational acceleration. These two forces combine to yield a downward acceleration a given by
>
> $$ma = mg \sin \theta - \mu \, mg \cos \theta.$$
>
> Using this value for a, the time required to reach the bottom can be found using the relation
>
> $$l = \tfrac{1}{2}at^2,$$
>
> Then using the known values of a and t, the block's final speed v is given by
>
> $$v = at.$$

The solution to a problem like this can be viewed as a sequence of physics principles selected and applied to the problem situation so as to generate new information, ultimately the information requested by the problem statement. We use the word *principle* generically to refer to any relation applied in the same form by most solvers, even though some solvers may relate a derived principle to more primitive ones. For example, we use the term principle for the expression $F_g'' = mg \sin \theta$ for the components of the gravitational force parallel ($''$) to a plane with inclination θ. A list of the principles used by the models is given in Table 3-3. We use *equation* for principles instantiated in a particular problem context. For example, the principle

$$x = v_0t + \tfrac{1}{2}at^2$$

is instantiated in the problem described above as the equation

$$l = \tfrac{1}{2}at^2$$

2. THE SIMULATION MODELS

2.1 Structure

The basic issues in the design of the simulation models are how to represent relatively permanent knowledge already available to the solver and how to represent the problem currently being processed.

Long-term memory, which holds the model's knowledge of physics and algebra, is represented as a collection of productions, each consisting of one or more conditions and one or more actions. The simulation models operate by testing these conditions, specifically by testing for the presence or absence of particular information (elements) in a limited-capacity *working memory*. When the conditions of a production match a set of elements in working memory, then its actions are executed, resulting in information being deleted from or added to working memory, or perhaps written onto an external *paper* memory that includes the kinds of things human solvers write on paper. Periodically a production causes some of the contents of the paper to be read into working memory. The effect is that important results are not irretrievably lost from working memory with the passage of time. Instead, working memory is regularly refreshed with important prior results.

These models are thus productions systems. They execute the actions for a sequence of productions, choosing productions in response to the contents of a relatively small working memory. This type of model is both conceptually simple and plausibly consistent with what we know about basic features of human information processing.

The models are programmed in a computer language called OPS2 (McDermott & Forgy, 1978; Forgy & McDermott, 1978). Since the aspects of OPS2 relevant to our simulation are described explicitly in this article, the reader will not need a knowledge of this language.

2.1.1. Working Memory. The working memory is small, typically containing about 20 elements. While this is not an exact match to "7 plus or minus 2," the hypothetical limit to the capacity of human short-term memory, 20 is not an implausible size, given the uncertainty about how information should be packaged into elements (chunks).

Examples of information stored in working memory include the current goal (e.g., to interpret symbols in a particular equation), the status (e.g., known, desired) of various quantities, and the assignment of symbols to quantities (e.g., t is the symbol for the time interval between instants 1 and 2).

2.1.2. How Productions Work. As outlined above, each production consists of two parts, a set of conditions (patterns that can match collections of elements in working memory), and a set of actions to be taken when the conditions are satisfied.

For example, consider the following production, written in English, except that, as in OPS symbols beginning with "=" are variables.

IF	the time interval between instants =x and =y is specified
THEN	assign the symbol t (=x, =y) to that interval, and associate with that symbol the status known.

If in working memory there are elements indicating that there is a time interval, then this production deposits in working memory an appropriate algebraic symbol for the interval, and associates with it the status known.

2.1.3. The Paper Memory. When new information is generated (e.g., a new interpreted equation, a new known quantity), that information is added to the separate paper memory. Intermediate results (e.g., the initial statement of an equation, a partially solved equation) are not put in the paper memory. Periodically (and always when no other productions are satisfied) the models read the paper memory into working memory, with most recently added entities becoming the most recent elements in working memory. Because important information is rarely irretrievably lost, and a summary of all past important information is regularly re-entered into working memory, these models have extra resilience to recover from temporary blind alleys. For example, if an equation is not interpretable because crucial information has been lost from working memory, this information is often recovered when the paper memory is next searched.

2.1.4. Selection of Productions. Commonly more than one production at a time has its conditions satisfied. In that case OPS resolves their conflicting claims according to the following guidelines (McDermott & Forgy, 1978). (1) A production cannot execute twice on the same data; this ensures that the system will not endlessly reconsider the same information, but rather will, if necessary, make use of all of the information available. (2) Preference is given to productions satisfied by the elements most recently placed in working memory; the effect of this rule is to predispose the system to continue to pursue whatever goal it is pursuing until it achieves the goal or determines that it cannot be achieved. (3) If one of a pair of satisfied productions is more specific than the other (i.e., matches a superset of the elements matched by the other), this more specific production is preferred; thus whenever the system has knowledge tailored to the current situation that knowledge is used instead of (or at least before) its more general knowledge.

The OPS interpreter automatically compiles the various productions into a net that discriminates, on the basis of features of a working memory element, those productions that could be satisfied. Then, during each recognize-act cycle, instead of sequentially testing the contents of working memory against each production (a lengthy and psychologically implausible process), the working-memory elements are efficiently sorted through the already-existing net (Forgy, 1979).

Although a major virtue of a production system is to allow powerful broad-band pattern matching, with potential access to any part of the model's knowledge at any time (Newell, 1973), in practice this effect is difficult to achieve, because if the pattern matching is kept truly open, then the model's attention is likely to wander badly and never build up new knowledge in a coherent way. A common solution to this problem is to package productions by putting among the conditions of each production an element indicating the goal to which it is relevant. Then, once a goal has been set, only productions with goals identical to that goal can execute. The result is better focus of attention but with the cost of losing broad-band access to the system's information.

To allow broad-band pattern matching with coherent attention, we have packaged productions using goals, but have retained a special goal (open), which allows the system periodically to engage in free-pattern matching. When the goal is open, then any production in the system can execute (assuming its other condition elements are satisfied). But once some production does execute, it changes the goal from "open" to some specific goal to which it belongs. Thereafter, only other productions associated with that goal can execute. When no such productions are satisfied, the goal is reset to open.

2.1.5. Problem Representations. Problems are given to the models as list structures in the paper memory. The elements of the initial problem representation include objects, contacts between these objects, and instants and intervals of time. Attributes of these entities specified by the problem (e.g., that at instant 1, object B has a known speed) are also indicated. For example, the following shows for the problem given in Section 1, the problem representation given to the models, stated in rough English rather than in OPS.

PROBLEM L1
MASS OF OBJECT B IS SPECIFIED
OBJECT B IS A BLOCK
OBJECT R IS A RAMP
DIRECTION OF CONTACT B R IS DOWN
COEFFICIENT OF FRICTION OF THE B-R CONTACT IS SPECIFIED
ANGLE OF OBJECT R IS SPECIFIED
SPEED AT INSTANT 1 OF OBJECT B IS SPECIFIED
SPEED AT INSTANT 2 OF OBJECT B IS DESIRED
DISTANCE DURING INTERVAL 1 2 OF OBJECT B IS SPECIFIED

The problem representation given to our models differs from the problem representation given to human solvers in that the natural-language statement has been coded, and the useful information organized by means of tags. This coding is analogous to what solvers do in constructing a labeled diagram such as Figure 1-1. In fact to translate from the human solver's sketch (Figure 1-1) to the preceding list representation would require a visual interpreter with the following capabilities: (1) Recognizing object types (e.g., block, ramp) from the shapes in a stereotyped drawing, (2) Noting when two objects are in contact, and the nature of that contact (e.g., surface, attached). (3) Associating information written on the diagram with objects and contacts and with various instants and intervals of time, (e.g., initial and final speeds with two instants 1 and 2, and distance with the time interval between those instants).

We have thus skipped the natural-language and image processing required for a model actually to read a problem statement and to construct such a representation for itself. We have done this both because we preferred to concentrate on subsequent phases of problem solving, and because this phase has been delineated by other work (Novak, 1976; Novak, 1977), in a system which translates natural language statements into descriptions of stereotyped objects, which are then used by the model to solve the problem. Another comparison with Novak's ISAAC system is in order. ISAAC's problem representations are basically geometric, with precise positions, orientations, and points of contact for all objects. In contrast, our models use a topological representation, with only contacts between objects specified. Part of this difference may arise from the difference in the types of problems solved by the two systems: lever problems for

ISAAC and moving object problems for our models. However, we find the sparser representation adequate for our purposes, and more plausible in terms of humans ability to store information.

2.2 Knowledge of Physics and Algebra

The models have memory elements (productions) with the following functions: (1) assigning symbols to aspects of the problem description, (2) selecting relevant physics principles, (3) generating the corresponding equation, (4) connecting symbols in an equation with information in the problem, and (5) solving equations.

2.2.1. Assigning Symbols. The models begin work by assigning an appropriate symbol to each description of a known or desired quantity in the problem; i.e., the models associate each description with a symbol that will match a symbol appearing in relevant equations. For example, given a time interval and an object's velocity at the beginning of that interval, a human assigns to it a symbol like v_0. Thus for each symbol the model must have knowledge of the kind of descriptions that can be assigned to it. This knowledge is embodied in productions like the example described in Section 2.1.2.

2.2.2. Selecting Equations. The selector productions select the principle to be considered next. Thus the action sides of these productions change the current goal to that of stating and interpreting some particular principle. The condition sides of the selectors constitute the heart of the models' strategies. This is where the information for deciding what to do next lies. For this reason, the nature of the selectors varies in the two models that represent different levels of skill in physics, variation that will be discussed in Section 2.3 on strategic knowledge.

2.2.3. Generating Equations. For our models "writing an equation" consists of adding to the paper (and working) memory elements representing each of the variables in the equation, with subelements indicating the equation to which these variables belong and the fact that these variables are currently not connected (bound) to any information in the problem.

Equations are always written for a particular physical context (system of objects, time interval, or both). For example, kinematics equations always refer to motion during a particular time interval. If an object is specified at all, then it must be the same object associated with each quantity. However, no object need be specified. Thus if a part of the physical context (e.g., what object is referred to) is omitted, the system simply assumes that all quantities refer to the same (unspecified) object.

2.2.4. Connecting Variables to Information in the Problem. When an equation is written, its variables are marked as "unbound," i.e., not connected to other information in the problem. The knowledge for making such connections is contained in the following single production.

IF there is an unbound algebraic symbol that is part of an equation and the same symbol, outside of any equation, has a status known (or desired).
THEN change the status of the variable in the equation to known (or desired).

Thus in its final form, as it appears in the paper memory, an equation has most of its variables marked as known or desired, with perhaps a few remaining unbound.

2.2.5. Solving Equations. In its present form, the models do not actually manipulate equations algebraically. For purposes of these simulations, it is necessary for them only to be able to make inferences about what information can be obtained from an equation. Hence, we model only the results of carrying out the details of the algebra, not the process. One production for "solving" equations recognizes when all but one of the variables in an equation are known, and asserts that the remaining variable is known. Others note when a previously desired quantity is known, and assert this quantity as an answer. There is one production that halts processing when *all* previously desired quantities are known and stops the problem-solving process.

2.3. Strategic Knowledge

In addition to the knowledge of physics and algebra described above, the models have strategic knowledge to enable them to decide when to do what. In the two models, we have implemented two strategies: (1) *Means-ends analysis* begins with the desired quantity and looks for equations including that quantity. Then it works backward, marking as desired any unbound quantity needed to solve such an equation. (2) *Knowledge-development* (forward chaining) begins with the known quantities in the problem statement and applies appropriate equations to derive new quantities from them until the desired quantity is reached. These strategies are implemented in the condition sides of the selector productions (Section 2.1.4).

2.3.1. Means-Ends Analysis. This model works in a backward, means-ends mode, employing the simple strategy of assessing the difference between the current problem state (the current equation) and the equation that will yield the desired answer, and then taking steps to reduce this difference. This strategy is determined by the selectors, which never propose a principle unless one of its variables is a desired quantity. In addition, if one quantity in a proposed equation remains unbound, a production marks it as a desired quantity, allowing the selectors to propose new equations involving it. [Generally the model simply abandons an equation for which more than one variable remains unbound, although in some circumstances, to be discussed later, this criterion is relaxed (see Section 4.1)].

This strategy produces problem solutions in which each new equation contains a variable marked as desired in the preceding equation. When the strategy succeeds in solving an equation, the value obtained is substituted into the preceding equation. Thus the model basically first works down, writing new equations for values to be substituted into earlier ones. Then it works upward, using values found from the later equations to bind variables in the earlier equations. The process has some parallels in de Kleer's NEWTON, which scans an equation and returns "complaints" of uninstantiated variables (de Kleer, 1975). NEWTON, however, has much more sophisticated capabilities for planning and implementing the resolution of such complaints.

This means-ends, algebraic search of principles is also used by MECHO, the physics problem-solver developed by Bundy (1975) although MECHO has considerable "meta-level" control (Bundy, Byrd, Luger, Mellish, & Palmer, 1979). MECHO's work to the principles chosen by novice solvers is discussed by Luger (1979).

The goals used to constrain attention (see Section 2.1.4) are associated with a particular equation. When the goal is open, the model uses its selectors to generate any appropriate equation. As the equation is written, the goal is changed from open to developing that equation, and the model is then constrained to work on that equation until nothing more can be done.

*2.3.2. **Knowledge Development.*** The model with this strategy uses a bottom-up, forward-working method. Its principle-selection mechanism is merely to note what values for variables are known, and to select a principle which allows finding the value of a new related variable. Thus if the model knows values for v, v_0 and t, it may select the principle

$$v = v_0 + at$$

so as to find the value for *a*. The result is a problem solution which is a series of principles, each applied to produce immediately new information from available information. As in the means-end model, strategic knowledge resides in the selectors that choose principles for application.

In addition to the preceding difference in strategy for selecting principles, our two models differ in the way they apply a selected principle. Recall that the means-ends model initially states a principle without knowing how (or whether) the variables involved can be connected to the variables specified in the problem. Thus this model must then explicitly collect the information to determine whether variables are known or desired. In contrast the knowledge-development model selects a principle only when it knows all but one of the quantities are known. Thus it is possible (although not necessary) to collapse or "automate" (Schneider & Shiffrin, 1977; Shiffrin & Schneider, 1977) the process of applying a principle by having the same production that selects a principle also apply it to state that the one unknown quantity can now be considered known. This automation is implemented in our knowledge-development model. Then what would otherwise be several steps of writing an equation, binding variables, and solving it, becomes just a single step of applying a principle to develop new information from known information. As we shall discuss later, these collapsed productions, which both select and apply principles in one step, are consistent with expert subjects' tendencies simply to state new results, often without the separate steps of stating an equation and explicitly connecting its variables to information in the problem.

Both differences (strategy and automation) between the means-ends and knowledge-development models result in less work for the latter model. First, a principle is never evoked unless it can be used, at least to generate *some* new piece of information; and in these simple problems, in fact, every equation generated by this criterion does ultimately contribute to the solution. Second, equations need not be written and then variables individually connected to information in the problem, but instead a new piece of knowledge is developed in one step. Thus solutions of the knowledge-development model are about one-half to two-thirds as long (in number of productions executed) compared with the corresponding solutions produced by the means-ends model.

The preceding mechanism works well in a limited domain such as straight-line kinematics (see Simon & Simon, 1978). However, to make the model work with problems from more diverse domains requires some means of constraining attention. To provide such focus, we have used the colloquial wisdom of physicists that the principles in their discipline are divided into methods or approaches, each containing something under five principles. The methods used by our knowledge-development model include kinematics (principles (5) through (7) in Table 3-3), forces (principles (1) through (4)) and work-energy (principles (7) and (8)). These methods seem closely analogous to the "RALCM's" of de Kleer (1975), which also group together a set of principles, and require that they be used exhaustively before new principles are invoked. To reflect this division of knowledge in the model, we use goals as described earlier (Section 2.1.4). Thus initially the model freely selects what principle to propose, with complete access to all its principle-proposing knowledge. However, once it has begun to work with a principle from the method of forces, it persists with that method until nothing further can be done.

2.3.3. Limitations in Strategy. The preceding strategies are very simplistic and limited. In one dimension, they do not allow for planning or qualitative reasoning, to mention two higher-order skills that many human solvers use. In a second dimension, the strategies are mathematically primitive, for example, they cannot (in the form described above) handle simultaneous equations. The goal of these models is to capture as simply as possible knowledge that is sufficient to locate and apply physics principles so as to solve a class of textbook mechanics problems in a manner consistent with that used by more and less skilled human solvers. We address later how these models may relate to important additional knowledge of human solvers (Section 5). The addition of further mathematical capabilities (simultaneous equations) is addressed in Section 4 on extensions of the models; such knowledge is often not possessed by novice solvers, and so does not appear in our basic models.

3. THE MATCH WITH DATA

3.1. Methodology

The main issue addressed in the remainder of this paper is to what extent the simulation models described here capture the problem-solving behavior of individuals with varying amounts of experience, working in several domains of physics. The human data matched against the simulation models comes from two sources: First, two subjects, one expert and one novice, worked 19 problems in straight-line kenematics (Simon & Simon, 1978). These data give an in-depth picture of the problem-solving techniques of two individuals on a substantial number of problems in a relatively limited domain. Second, eleven expert and eleven novice subjects solved two problems in dynamics. This study gives a somewhat broader picture of how experts and novices work, without the in-depth study of any particular individual. In addition, these problems are more difficult, requiring knowledge of more physics principles and offering more alternatives in solution method.

3.1.1. Think-Aloud Protocols. In both cases, data were collected by presenting a subject with the problems and asking him to "think aloud as much as possible while working the problems." Experimenter intervention was kept to a minimum. With unpracticed subjects (Dynamics), the experimenter said only such things as "Can you say what you're thinking" after a long pause. Practiced subjects worked alone, talking into a tape recorder. The tapes were then transcribed verbatim.

The initial coding of the transcribed protocols consisted of simply listing from the taped transcript every quantitative relation mentioned. These included statements of principles (e.g., "F equals m a"), instantiations of principles (e.g., "The force of the spring is point three times fifty"), algebraic combinations, and statements of values (e.g., "m equals 5 kilograms"). This general list was fairly easy to make, and an unskilled coder could do much of it.

From this uncritically assembled list we then selected statements reflecting processes of interest to us. First, we constructed a sequential list of each physics principle mentioned or implied in a protocol statement. The novice sequences were then further edited, deleting earlier mentions of a principle stated but not used until later in the solution. The reason for these deletions is that novices often state a number of principles early in the solution, and then never do anything with them; they then restate the same principles later when they actually are used. [Current work is beginning to suggest that these early statements of principles reflect incomplete and unsuccessful use of a strategy similar to that used by more competent solvers (Larkin, 1981).]

3.1.2. The Kinematics Study.

Subjects. The expert subject had strong mathematics skills and extensive experience in solving problems in mechanics. The novice had fair skill in algebra, but had studied a chapter on kinematics only recently for the purposes of this study.

Problems. The 19 problems could all be solved by combined application of the princples for straight-line kinematics with constant acceleration listed in Table 3-1.

Most of the problems, as illustrated by the following, required application of one or two principles to a single context (object and time interval).

> A car traveling at 25 m/sec is brought to rest at a constant rate in 20 sec by applying the brake. (a) What is its constant acceleration? (b) How far did it move after the brake was applied?

3.1.3. The Dynamics Study.

Subjects. The novice subjects were enrolled in their first university-level physics course, a course requiring calculus and designed for engineering and physics majors. They had completed about eight weeks of work, including study of the kinematics, force, energy, and work principles relevant to the problems presented. The expert subjects were professors and advanced graduate students who had within two years been involved in teaching the material relevant to the problems presented. The data for three of the eleven novice subjects have been omitted because they were so chaotic as to elude analysis for the present.

Problems. Both problems, (see Table 3-2) require fairly straightforward applications of principles of either energy and work or forces with kinematics, applied to just one context.

TABLE 3-1
Principles Used by the Models in Solving Kinematics Problems.

(1) $\bar{v} = x/t$
(4) $v = v_0 + at$
(5) $\bar{v} = (v_0 + v)/2$
(7) $x = v_0t + \frac{1}{2}at^2$
(8) $v^2 - v_0^2 = 2ax$ (novice model only)

where x is the distance traveled by an object during a time t, with constant acceleration a, initial speed v_0, final speed v, and average speed $\bar{v}$.

The nonsequential numbering is chosen for consistency with an earlier report of these data (Simon & Simon, 1978).

TABLE 3-2
Problems Used in the Dynamics Study.

Problem 1. A block of mass m starts from rest down a plane of length l making an angle θ with the horizontal. If the coefficient of friction between block and phane is μ, what is the block's speed as it reaches the bottom of the plane?

Problem 4. What is the minimum stopping distance for a car travelling along a flat horizontal road if the coefficient of friction between tires and road is μ

For example, the solution to problem 1 in Table 3-2 corresponds to applying the sequence of principles for forces and kinematics listed in Table 3-3.

3.2. Tailoring the Models

After the models were implemented as described earlier, a few changes, described in detail in the following paragraphs, were made to enhance the fit of the output of the models to the observed data.,

TABLE 3-3
Principles Used by the Models in Solving Dynamics Problems

Force and Kinematics

(1) $F_g'' = mg \sin \theta$
(2) $f = \mu N$
(2') $N = mg \cos \theta$
(3) $F = \sum F's$
(4) $F = ma$
(5) $x = v_0t + \frac{1}{2}at^2$
(6) $v = v_0 + at$,
(7) $v^2 - v_0^2 = 2ax$ (novice only)
where x, v, vo, a and t are as defined for the kinematics problems (Table 3-1), F_g'' is the component of the gravitational force acting parallel to a plane and f and N are the frictional and normal forces exerted by the plane on an object resting on it.

Work and Energy

Principles (1) through (3) above, together with:
(7) $W = Fx$
(8) $K = \frac{1}{2}mv^2$,
(8') $K_f - K_i = W$
Where K_i and K_f are initial and final kinetic energies, K is either the initial or final kinetic energy, and is related to the corresponding speed v. In place of principle (7), some novice subjects used the following principles, together with (1) through (2') above:
(10) $W = \sum W's$
(11) $W_g = mgh$
(12) $h = x \sin \theta$
(13) $W_f = fx$,
where W_g and W_f are the individual works done on an object by the frictional and gravitational forces, and h is the height through which an object moves.

In the basic means-ends model, an equation is written only if it contains a currently desired quantity. This constraint by itself would often allow several possible equations to be written. To further specify the conditions under which equations are written, and to fit better the data from our novice subjects, we added the following selection rule. If there is more than one equation containing the desired quantity, then select the principle containing more known quantities. For example, if v is desired

$$v^2 - v_0^2 = 2ax$$

is written if x is known, while

$$v = v_0 + at$$

is written if t is known.

In basic knowledge-development simulation an equation is selected only when all but one of the quantities in the equation are known. As in the means-ends case, if more than one equation satisfies this criterion, then we have designed selectors which discriminate further in a manner consistent with the work of our human subjects. Specifically, when a and t are known, the equation

$$x = \tfrac{1}{2}a\, t^2$$

is used for an object falling or rolling from rest, while in other situations,

$$v = at$$

is used. If there is still more than one possible equation, then the model prefers one containing the desired quantity.

In two cases we have further collapsed of productions to reflect better the work of our expert subjects. In applying the common pair of principles describing the normal force N and the relation between it and the frictional force f, skilled solvers almost always either state a combined relation, or state the two very close together and in variable order. Thus in our knowledge-development model, one production applies the combined version of these principles, and for this model we use the tag (2) (see Table 3-3) to refer to the combined principle $f = \mu mg \cos\theta$. Similarly, in the two problems considered here, the total work W done on a particle is equal to its non-zero initial or final kinetic energy K. Experts use these quantities interchangeably, and so we have not distinguished between them in the knowledge-development model, referring to them both by (8) in Table 3-3. This collapse might be considered a further instance of the kind of automation of processing discussed in Section 2.3.2. When a group of steps are often performed in sequence (e.g. stating principles and instantiating variables or applying two related principles) by in a more competent solver, the process can be automated into a much shorter combined process.

Finally, for the dynamics problems, both models produce different correct solution paths depending on the order in which the principles in Table 3-3 are searched. If force and kinematics principles are searched first, the solution path involves the first group of principles in Table 3-3. If work-energy principles are searched before kinematics principles, then the solution involves principles (1)–(3) and (7)–(8). For problem 4 in Table 3-2, there are two work-energy solutions paths for the KD Model, one reflecting first a search of work and energy principles, followed by force principles, and the other reflecting a search of force principles (1) through (4) followed by work and energy principles. The final set of principles can also be used. These four kinds of solution paths were all produced by the models and were available for matching against the human solvers' work.

3.3. Order of Application of Principles

Our major test of the simulation models is the extent to which they apply principles in an order similar to that used by human solvers.

3.3.1. Kinematics. Table 3-4 summarizes the work on the 19 kinemátics problems for the two human solvers and for the knowledge-development (KD) and means-ends (ME) models. The left column lists the variables given (G) and desired (D) in the problem, together with the problem number (#). In the remaining columns principles are listed in the order in which the solvers introduced them. The letters indicate the variable solved for and the numbers the kinematics principle introduced (listed in Table 3-1). Thus v4 means that principle 4,

$$v = v_0 + at$$

was solved for v. The problems are listed in an order that groups together problems involving the same known or desired variables.

The good match between the human and simulated data essentially replicates earlier work (Simon & Simon, 1978), with a somewhat better correspondence due to the tailoring of the models described in Section 3.2. First the KD model discriminates on the basis of desired quantities if the known quantitities satisfy more than one equation, and the ME model discriminates on the basis of known quantities if the desired quantity appears in more than one equation. Thus the KD system and the expert subject solve problems 7 and 17 differently from problems 23 and 25. Similarly the novice subject and ME model select different equations involving v in problem 16 and in problem 5.

Second, the KD model uses some more information from the problem representation. Specifically, this model, like the human expert, uses equation 7,

$$x = \frac{1}{2}at^2,$$

for problems involving objects falling from rest (problems 11 and 12 in Table 3-4), while continuing to use other patterns for other situations with identical known quantities (problems 8-10 and 20-21).

Third, the order of principle application for the novice is somewhat different from that reported in Simon and Simon (1978) because that paper reported

TABLE 3-4
Order of Principles Applied for Kinematics Problems.

Problem G	D		Expert Protocol	KD Model	Novice Protocol	ME Model
v_0at	vx	5	v4-x7	v4-x7	v4-x7	v4-x7
v_0at	vx	8	v4-$\bar{v}$5-x1	v4-$\bar{v}$5-x1	v4-x7	v4-x7
v_0at	vx	9	v4-$\bar{v}$5-x1	v4-$\bar{v}$5-x1	v4-x7	v4-x7
v_0at	vx	10	v4-$\bar{v}$5-x1	v4-$\bar{v}$5-x1	v4-x7	v4-x7
v_0at	xv	11	v4-x7	v4-x7	v4-x7	v4-x7
v_0at	x	12	x7	x7	x7	x7
v_0at	x	20	v4-$\bar{v}$5-x1	v4-$\bar{v}$5-x1	x7	x7
v_0at	x	21	v4-$\bar{v}$5-x1	v4-$\bar{v}$5-x1	x7	x7
v_0vt	x	23	$\bar{v}$5-x1	$\bar{v}$5-x1	x7	x7
v_0vt	x	25	$\bar{v}$5-x1	$\bar{v}$5-x1	x1-v5 (x7-a4)[b]	x1-v5 (x7-a4)[b]
v_0vt	ax	7	a4-$\bar{v}$5-x1	a4-$\bar{v}$5-x1	a4-x7	a4-x7
v_0vt	ax	17	a4-$\bar{v}$5-x1	a4-$\bar{v}$5-x1	a4-x7	a4-x7
v_0va	tx	6	t4-$\bar{v}$5-x1	t4-$\bar{v}$5-x1	t4-x7	t4-x7
v_0va	tx	18	t4-$\bar{v}$5-x1	t4-$\bar{v}$5-x1	x1-t4-$\bar{v}$5 (x7)[b]	x1-$\bar{v}$5-t4 (x7)[b]
v_0vx	at	24	[a]	v5-x1-a4	a8-t7	a8-t7
v_0vx	vt	19	$\bar{v}$5-t1	v5-t1	v5-t1	v5-t1
v_0ax	vt	16	t7-v4	t7-v4	v8-t7	v8-t7
v_0xt	$\bar{v}$v	13	a7-v4-$\bar{v}$5	a7-v4-$\bar{v}$5	v8-a7-$\bar{v}$5	a7-v8-$\bar{v}$5
st	$\bar{v}$	3	$\bar{v}$1	$\bar{v}$1	$\bar{v}$1	$\bar{v}$1

[a]Anamalous solution (Simon & Simon, 1978).
[b]Corrections following erroneous statements (see text).

the final order in which principles were used, whereas here we have traced the backward generation of principles.

The current models account for 18 out of 19 or 94 percent of the expert solution paths and 17 out of 19 or 89 percent of the novice paths. In contrast, the KD model matches the novice solution on 4 out of 19 problems and the ME model matches the expert solution on 5 problems.

In addition, we succeeded in matching some errors and recoveries of our novice solver by simply deleting the output of a production corresponding to an equation miswritten or apparently forgotten by the human subject. Specifically, in problem 25, the novice solver starts by trying to solve

$$(1)\ \bar{v} = x/t$$

for the desired quantity x. Not having $\bar{v}$, she tries to solve

$$(5)\ \bar{v} = (v_0 + v)/2,$$

for $\bar{v}$, but becomes confused. She then starts over, trying to solve

$$(7)\ x = v_0t + (½)a\ t^2$$

for x, and then

$$(4)\ v = v_0\ at$$

for a. By deleting the output of the production producing equation (5), the equation she became confused about, we caused the ME model to replicate the two principles used in her alternate solution. Similarly, for problem 18, deleting the output of the production producing principle (5) caused the ME model (like the novice solver) to recover by using instead principle (7) to solve for the desired quantity x.

3.3.2. Dynamics. Tables 3-5 and 3-6 show for the dynamics problems in Table 3-2 the order in which principles were applied by the expert subjects (indicated by subject numbers) and by the knowledge-development simulation model (indicated by KD). Each principle is represented by its number (see Table 3-3). The first quantity indicated is the quantity for which this principle is solved, and, if all other quantities in the principle are known, this is the only quantity that appears. Thus for example, the principle:

$$(1)\ F_g'' = mg \sin \theta$$

where m, g, and θ are known, is represented by $F_g'' - 1$. If some other variable in the principle is not known, it is indicated in parentheses. For example, v(t) – 6 indicates an expression for v was written, using equation (6), and involving the unknown quantity t. As described earlier (Section 3.2), expert subjects (and our KD model) do not use separately equations (2) and (2′) or equations (8) and (8′) (see Table 3-3), and in these cases the unprimed number refers to the joint application of the two principles. The symbol (–) indicates a principle not explicitly stated but used implicitly to find a subsequent quantity listed in the table. For example, in Table 3-5, subject 1 does not explicitly state an expression for F, but uses it in constructing the subsequent expression for K.

Comparing the order of principle application for the KD model and for the expert subjects, interchanges of two principles are common (subjects 3, 5, 7, 8, and 9 in problem 1; and subjects 2 and 10 in problem 4). In addition the experts sometimes do several steps together, as indicated by dashes (–) in the tables. Beyond these deviations, the KD model fails to account for the order of principle application for 2 out of the 21 problems solved. First, in both problems, subject 11 used unusual principles to solve for v or x,

$$v^2 - v_0^2 = 2ax$$

in problem 1, and

$$\bar{v} = (v + v_0)/2$$

and

$$x = \bar{v}t$$

in problem 4. These solutions are consistent with the knowledge-development strategy, but reflect a different order of searching for principles.

Thus the experts essentially always worked forward, generating new information, in an order consistent with that produced by the knowledge-development model. The major exceptions are combining several principles into one step (notably subject 4 in both problems), and interchanging two steps (once for about half the subjects on each problem).

Tables 3-7 and 3-8 show the order in which principles are applied by novice subjects and by the means-ends (ME) simulation model in solving the problems in Table 3-2. Each principle is represented by its number (see Table 3-3), by the variable for which it is solved, and by the independent variables that are unknown at the time it is evoked. Thus v(a,t)−6 represents the principle

$$(6)\ v = v_0 + at$$

evoked to solve for v when v_0 is known and a and t are not. As in Tables 3-5 and 3-6, a variable alone (e.g., F_g) indicates a principle used to find that variable in terms of known quantities, and "−" indicates a principle used implicitly but not stated explicitly. Again both force-kinematics and two work-energy solutions are shown.

TABLE 3-5
Order of Principles Applied by
Expert Subjects and KD Model on Dynamics Problem 1.

				Force-Kinematics Solutions				
Solver:	KD	1	3	5	6	7	8	11
1st	F_g''-1	F_g''-1	F_g''-1	f-2	F_g''-1	F_g''-1	F_g''-1	F_g''-1
2nd	f-2	f-2	f-2	F_g''-1	f-2	f-2	f-2	f-2
3rd	F-3	—	—	—	—	—	—	—
4th	a-4	a-4	a-4	a-4	a-4	a-4	a-4	a-4
5th	t-5	t-5	v(t)-6	v(t)-6	t-5	v(t)-6	v(t)-6	v_o
6th	v-6	v-6	t-5	t-5	v-6	t-5	t-5	

		Work-Energy Solutions		
Solver:	KD	4	9	10
1st	F_g''-1	F_g''-1	f-2	F_g''-1
2nd	f-2	f-2	F_g''-1	f-2
3rd	F-3	—	—	F-3
4th	W-7	v(W)-8	W-7	W-7
5th	v-8	W-7	v-8	v-8

Known quantities: m g θ μ v_0 x
Desired quantity: v
Subject 2 missing due to tape failure.
°Unusual principle, see text.

TABLE 3-6
Order of Principles Applied by Expert Subjects and KD Model on Dynamics Problem 4.

Force-Kinematics Solutions

Solver:	KD	1	3	5	8	11
1st	f-2	f-2	f-2	f-2	f-2	f-2
2nd	F-3	—	—	—	—	—
3rd	a-4	a-4	a-4	a-4	a-4	a-4
4th	t-6	t-6	t-6	t-6	t-6	t-6
5th						v^a
6th	x-5	x-5	x-5	x-5	x-5	x^a

Work-Energy Solutions

Solver:	KD	2	4	9	KD[b]	10[b]	6[b]	7[b]
1st	f-2	—	—	f-2	W-8	f-2	W-8	W-8
2nd	F-2	—	F-2	—	f-2	W-8	—	x(F)-7
3rd	W-8	x(W)-7	W-8	W-8	F-3	F-3	F-3	F-3
4th	x-7	W-8	x-7	x-7	x-7	x-7	x-7	

Known quantities: m g μ v_0 v
Desired quantity: x
[a]Unusual principle, see text.
[b]Different search order, see text Section 3.2.

As Tables 3-7 and 3-8 show, the novice solvers consistently start by working backward, i.e. with principles involving the desired quantity, and their order of principle application is roughly consistent with that produced by the ME model, with the following main exceptions:

First, the ME model, like subjects 2, 4, and 8, has available two principles for expressing v in problem 1 (or x in problem 4), (7) and (6) (or (5)), both involving acceleration a and (6) and (5) involving in addition time t (see Table 3-3). The ME model first tries principle (6) for v or (5) x involving a and t (the more commonly used principle). Finding it contains two variables that cannot be connected to information in the problem, it abandons it and generates principle (7) (see Section 2.3.1). In contrast, in both problems, subject 1 uses only the first principle ((6) or (5)), ultimately finding an expression for t from the other of principle (5) or (6), as does subject 4 in probelm 4. Subject 10 in problem 1 begins work with principle (7).

Second, the human novice subjects often write principles in nonstandard (or incorrect) form, indicated in the tables by variables without principle numbers. Most prominently, in problem 1 only subject 1 expresses acceleration in terms of total force F, using the standard equation

$$(4)\ F = ma.$$

TABLE 3-7

Order of Principles Applied by Novice Subjects and ME Model on Dynamics Problem 1.

Force-Kinematics Solutions

Solver:	ME	1	2	4	8	10
			F_g''-1			
earlier			f-2			
1st	v(a,t)-6	v(a,t)-6	v(a,t-6)	v(a,t)-6	v(a,t)-6	
2nd	v(a)-7	t(a)-2	v(a)-7	v(a)-7	v(a)-7	v(a)-7
3rd	a(F)-4	a(F)-4	$a(a_g,a_f)$	a(f)	a(N)	a(N)
4th	F(F_g'' f)-3	—	—	—	—	—
5th	F_g''-1	F/¼	AG(F_g'')			—
6th	f(N)-2	F_g''-1	a_f(f)	f-2		f-2
7th	N-2′	N-2′			N-2′	N-2′

Work-Energy Solutions

Solver:	ME	9	ME[a]	11[a]	6[a]
	v(K)-8				
1st	K(W)-8′	v(W)-8,8′	v(W)-8	v(W)-8	v(a,t)
2nd	W(F)-7	W(F)-7	$W(W_g,W_f)$-10	—	a
3rd	F(F_g'',f)-3	—	W_g(h)-11	W_g(h)-11	t(x)
4th	f(N)-2	f(N)-2	h-12		x
5th	N-2′	N-2′	W_f(f)-13	W_f(f)-13	
6th	F_g''-1	F_g''-1	f(N)-2	f(N)-2	
7th			N-2′	N-2′	
8th				h-12	

Known quantities: m g θ μ v_0 x

Desired quantity: v

[a]Solutions using final group of work principles, see Table 3-3.

Variables without principle numbers indicate unusual principles, see text.

The remaining subjects instead relate acceleration to various individual forces, e.g., friction *f* or normal force *N*. Subject 2 even relates the "total" acceleration *a* to two "individual" accelerations a_g and a_f due to the gravitational and frictional forces. Subjects 4, 8, and 10 have incorrectly related acceleration only to the frictional (or normal) force, and completely neglected the gravitational force. Similarly, in problem 4 subject 9 omitted the gravitational work W_g. Many of the equations appearing for subject 6 are completely erroneous equations, but used in a consistent means-ends pattern. Subject 11 relates W to N rather than F. In all these cases, modifying the ME model so that it can apply its means-ends strategy to these non-standard expressions allows the model to replicate the order of principles generated by the novice subject.

Finally, some novice subjects do begin their work by finding values for quantities in terms of quantities with known values, i.e., by what we call knowledge development. This is most evident for subjects 10 and 11 on problem 4. Indeed their work is very close to that the of the KD model shown in Table 3-6. But one or two quantities correspond to the KD Model for subject 2 in problem 1 and for subjects 1, 2, 4 and 8 in problem 4.

Table 3-8
Order of Principles Applied by
Novice Subjects and ME Model on Dynamics Problem 4.

Force-Kinematics Solutions

Solver:	ME	1	2	4
		f(N)-2	f(N)-2	f(N)-2
earlier		N-2'	N-2'	N-2'
				F-3
1st	x(a,t)-5	x(a,t)-5	x(a)-7	x(a,t)-5
		t(a)-6		t(a)-6
2nd	x(a)-7		x(a,t)	x(a)-7
3rd	a(F)-4	a(F)-4	a(F)-4	a/F)-4
4th	F(f)-3	—	—	
5th	f(N)-2	—		
6th	N-2'	N-2'		

Work-Energy Solutions

Solver:	ME	8	9	10[a]	11[a]
		N-2'			
earlier		f-2			
		F-3			
1st	x(W,F)-7	x(W,F)-7	x(W,F)-7	N-2'	
2nd	W(K)-8'	W(K)-8		f-2	W-8,8'
3rd	K-8	K-8'		W-8,8'	x(N)-7
4th	F(f)-3	x-7	F(f)-3		N-2'
5th	f(N)-2		N-2'		
6th	N-2'		f-2		

Known quantities: m g μ vo v
Desired quantities: x
[a]Expert-like solutions, see text.

Thus the ME model does not account for all the inaccuracies and variability of the various novice subjects on these relatively difficult problems. However, within these variations, the pattern of working backward, using a means-ends strategy, is consistent.

3.4. Relating Variables to Known Information

The second difference between the knowledge-development and means-ends models (in addition to different orders of principle application) is the way in which information in the problem is connected to variables in an equation. In the means-ends models, the equation is first explicitly written, and then each variable in it is explicitly "bound" to any known or desired variables from the problem. In the knowledge-development model, equations do not appear at all. Instead, the information about what independent variables must be known appears in the condition side of a production, and the value of the corresponding dependent variable which can be found is produced automatically by the action side.

Clearly these models are extreme. Skilled solvers do state equations (although not always). However, after writing an equation, these solvers rarely explicitly mention the values of its variables, but simply proceed to solve the equation, using these values. Thus apparently these solvers already know the values of the variables as they write the equation, and do not need separately to bind each variable to information in the problem.

As a criterion for whether a subject explicitly bound a variable, we looked for statements in the protocols of the form "*x* equals 5 meters," where "*x*" could be any variable appearing explicitly in the previously mentioned equation, and "5 meters" is the already known value for that variable.

Applying this criterion to the 18 kinematics problems worked by the individual expert and the novice, we found that the novice indicated binding variables a total of 24 times or an average of 1.2 bindings per problem. In contrast the expert bound variables a total of four times, all while using the equation $s = \frac{1}{2} at^2$, the one equation with which he seems particuarly uncomfortable (see Simon & Simon, 1978).

In the dynamics problems, worked by 8 novices and 11 experts, the average number of explicit bindings per problem was 0.1 for the experts and 0.9 for the novices.

4. EXTENSIONS

In addition to the account of the order in which principles are applied and the prevalance of explicit bindings among novices, we have extended our models in two ways. First we have exploited the flexibility of our representation to develop a completely analogous simulation model that solves problems in engineering thermodynamics in a manner consistent with a model and data described earlier (Bhaskar & Simon, 1977). Second, we have extended the simulation models described here, by giving them a capacity to discriminate between and to relate different contexts. These additions enable the models to solve a more difficult problem from kinematics, and they give further insights into the difficulties novice solvers encounter with this problem.

4.1. Flexibility of the Representation

The problem representation scheme described earlier for kinematics and dynamics problems is sufficiently flexible to represent thermodynamics problems such as the following:

> Steam enters the nozzle of a turbine with a low velocity at a pressure of 400 lbf/sq in. at 600 F. It leaves the nozzle at 260 psia at a velocity of 1540 ft/sec. The rate of flow of steam is 300 lbm-hr. Calculate the quality or temperature of the steam leaving the nozzle and the exit area of the nozzle.

This problem, like the others discussed originally in (Bhaskar & Simon, 1977), requires the solver to relate various state variables of some substance at the beginning and end of a process to various characteristics of that process. All are solved by applying conservation of mass and energy to relate variables at the beginning and end of the process, and applying various state equations (or tables of state variables) to relate the state variables of the substance at a single instant of time. Such problems can be represented by using instants and intervals in the way developed for kinematics and dynamics. For example, the following is a representation of the problem stated above.

OBJECT N is a NOZZLE
SUBSTANCE S is STEAM
VELOCITY at INSTANT 1 is KNOWN
PRESSURE at INSTANT 1 is KNOWN
TEMPERATURE at INSTANT 1 is KNOWN
PRESSURE at INSTANT 2 is KNOWN
VELOCITY at INSTANT 2 is KNOWN
FLOW-RATE during INTERVAL 1 2 is KNOWN
TEMPERATURE at INSTANT 2 is DESIRED
AREA at INSTANT 2 is DESIRED

To account for the observed human performance on such problems requires a modification of our means-ends model. When this model works in its original manner, the first equation proposed always involves one of the quantities requested as an answer (because these are the only initially desired quantities). We now modify the model such that the initial equation can be anything (here the principle of conservation of energy). Then typically many variables initially remain unbound, and we relax our earlier criteria that an equation be abandoned if more than one variable remains unbound. Thereafter the modified model works like the original model. All possible variables in the proposed equation are bound to information in the problem, the remaining variables are marked as desired, and selectors propose new equations involving these desired quantities.

How well does the preceding account for the order in which principles are applied by a human solver? The single subject reported (Bhaskar & Simon, 1977) was a teaching assistant in a self-paced course in chemical engineering thermodynamics. Thus he was reasonably proficient at solving the presented problems, although there may be some question as to whether he is truly an expert subject.

The analysis of data is best handled by comparison with the original analysis using the SAPA automatic protocol coder (Bhaskar & Simon, 1977). This coder allows entry of an initial equation (which was always some version of conservation of energy) and then systematically considers each unbound variable, developing an equation or a table look-up which would allow finding that variable. Thus our extended model makes predictions identical to those made by the SAPA model and thus also accounts for the data in the earlier report (Bhaskar & Simon, 1977).

4.2. Discriminating Context

All of the problems thus far discussed share the simplicity of containing only one context (i.e., one object and one time interval) to which principles can be applied. However, in many problems several contexts are involved, and to handle contexts well requires a representation that is not burdened with unneeded context information but that allows the use of this information when necessary.

The OPS Language has a facility for matching only as much of an element as is needed. Specifically, an element of the form

((object =x) . . . =y)

matches any element which begins with (object =*x*); the variable =*y* is bound to whatever follows. Productions containing such an element will match either of the following elements:

((object B))
((object B) (interval 1 2)).

In the first case, =*y* is bound to nothing, and action elements containing =*y* will simply not mention time intervals. In the second case, =*y* is bound to (interval 1 2), and action elements containing =*y* will specify this time interval. The effect is a single production that pays attention to additional context information (time) when it is mentioned, but does not require this information when it is not important.

The process for reading information from the paper memory also provides a mechanism for focusing attention on a particular context in a multiple-context problem. When one production has been activated with an element specifying a particular context [e.g., ((object B) (interval 1 2))], then the paper memory is read, looking first at elements having this same context. Thus such elements are kept most recent in working memory, and because of the conflict resolution rules, productions preferentially match these elements.

4.3. Relating Contexts

In addition to discriminating contexts, solving problems with multiple contexts requires knowledge about relating contexts. For example, consider the following problem:

> At the moment car A is starting from rest and accelerating at 4 m/sec^2, car B passes it, moving at a constant speed of 28 m/sec. How long will it take car A to catch up with car B?

The following additional knowledge is needed to enable either of our models to solve it correctly. (1) It must recognize that if two objects have the same position at instant 1 and again at instant 2, then they travel the same distance during the interval between 1 and 2. (2) It requires the ability to solve simultaneous equations, e.g., to write two expressions for a single quantity (here the distance x traveled by either car), where both expressions include the same desired quantity (here t), so as to obtain the equation.

$$v_B t = \tfrac{1}{2}at^2$$

which can be solved for t.

Table 4-1 shows the work of the knowledge-development simulation model with this information added, together with that of the expert subject. The major difference between the solutions is that the simulation, using the selectors described earlier, finds x_A in terms of a and t by first finding $\bar{v}_A$, while the human solver uncharacteristically uses the relation $x_A = \tfrac{1}{2}at^2$.

To account for the novice performance on this problem, we took the original means-ends simulation model and added to it the first piece of knowledge listed above but not the second. Then the model produces the work shown at the left in Table 4-1. Notice that in each equation there is one unbound variable, which is marked as desired, and which then prompts the generation of an equation involving it. The result is a circular pattern, which we ultimately terminated.

The work of the novice solver is also shown in Table 4-1. She applied principles in an order consistent with that of the means-ends simulation. However, as shown below the line, instead of wandering forever, she ultimately did remember or invent the strategy of substituting in an expression for a desired quantity a second expression involving the same desired quantity, and thus was

TABLE 4-1
Order of Principles Applied on a Problem Involving Two Contexts.

KD	*EXPERT*
$x_A = \tfrac{1}{2}at^2$	$v_A = at$
	$\bar{v}_A = v_A/2$
$x_A = \bar{v}_A t$	
$X_B = v_B t$	$x_B = v_B t$
$\tfrac{1}{2}at^2 = v_B t$	$\tfrac{1}{2}at^2 = v_B t$
t known	t known
ME	*Novice*
$t_B = x_B/v_B$	$t_B = x_B/v_B$
$x_B = x_A$	$x_B = x_A$
$x_A = \tfrac{1}{2}at^2$	$x_A = \tfrac{1}{2}at^2$
$t_A = t_B$	$t_A = t_B$
$t_B = x_B/v_B$	$t_B = x_B = x_B/v_B$
(circular)	$x_B = \tfrac{1}{2}a(x_B/v_B)^2$
	solve for x_B
	solve for t_B

able to solve for x, and finally for the answer t. In other words she used knowledge of how to relate two contexts by using simultaneous equations.

We think this rather ad-hoc treatment of simultaneous equations suggests an important aspect of human knowledge of algebra. As evidenced by students' common use of extensive algebra to prove "5 = 5", and by the analysis of the graph structure of how variables are related in kinematics done by de Kleer (1975), the knowledge required to handle simultaneous equations completely is enormous. Our suggestion is that instead skilled human solvers have heuristics for generating two equations describing two different contexts (objects or time intervals), the heuristic described above.

5. TOWARD MORE COMPLETE MODELS OF COMPETENCE

In this section we speculate on how the models described here might be expanded to provide a more complete explanation of competence in a domain of technical knowledge such as physics.

The major limitation we see in the current models is their use of an exceedingly primitive problem representation. In fact, after the initial problem representation, these models work only with algebraic quantities and principles described by algebraic equations. Thus in a sense they have no knowledge of physics, but only of the algebraic representation. This may actually not be too bad for capturing the performance of novice solvers, who have little knowledge of physics, but much more of algebra. However, it is certainly inadequate for capturing the work of more competent solvers.

We have described elsewhere (McDermott & Larkin, 1978) a set of problem representations that we think much better reflect representations used by skilled human solvers. The computer implemented problem solver (PH632) described in that earlier paper uses a sequence of four problem representations. The first is the verbal problem representation, as it might be stated in a textbook. The second is a "sketch" or "real-world" representation that involves real-world objects as they might be portrayed in a sketch of the problem situation. This representation used by the models described here and shown in Section 2.1.5. PH632 then develops a third "physical" representation of the problem, that contains, not real-world objects, but physics objects such as forces and energies. This physical representation is then translated into the equations that are used to solve the problem.

In future work, we think that the richer representation scheme of PH632, in particular the physical representation, should be combined with the knowledge-development capability of the models described here to build a more complete model of competent problem-solving behavior. We speculate that skilled human solvers are capable of automatically and easily developing knowledge about a problem situation, not just about quantities (as done by the models described here), but also about the physics entities used by PH632. The easy ability to generate knowledge enables the skilled solver readily to re-represent the problem (e.g. in terms of forces), and then to decide whether the generated knowledge about forces is sufficient to solve the problem. Because producing this information is easy, little is lost if this representation is abandoned and a new one (e.g. energies is produced). In a difficult problem, several of these physical representations may be generated before a specific approach is selected (Larkin, 1977). If the problems are sufficiently easy (like those described here) then the skilled solver can simultaneously generate both qualitative physical knowledge and quantitative knowledge sufficient to complete the solution of the problem.

This view of the importance of a physical representation also suggests one reason for the puzzling phenomena of the novice solvers, extensive use of the principle

$$v^2 + v_0^2 + 2ax$$

a principle almost never used by expert solvers. The terms in this principle, suggested by (Simon & Simon, 1978), correspond to nothing in a physical representation. In contrast, the principles that are used by expert solvers (e.g. $v = v_0 + at$, $x = v_0t + \frac{1}{2}at^2$) have terms which are transparently connected to physical features in the problem, distances and changes in distances, speeds and changes in speeds.

6. SUMMARY

We have presented a set of two working simulation models that capture features of different levels of competence in solving problems in elementary physics. These models are production systems that use an external memory in a manner analogous to a human solver's use of paper.

The first difference in the two models characterizing different levels of competence, is in their strategic knowledge—knowledge for deciding when to apply what principle. The means-ends model, corresponding to the work of many novice solvers, focuses on the quantity to be found, writes an equation involving that quantity, and then works backward, writing expressions for quantities that remain unknown. The knowledge-development model, corresponding to the work of our skilled solvers, recognizes patterns of information that allow development of a new piece of information.

The second difference in the models lies in the way in which principles are used. In the means-ends model, a selected equation is written, and then variables in it are individually connected to known or desired variables specified by the problem. In the knowledge-development model, the selection and application of a principle has been combined, allowing the model to collect necessary information and use it to generate new information all in a single step.

While these differences are fairly extreme characterizations of behavior, they do match quite well the order in which principles are applied by skilled and less skilled subjects working a variety of problems. Furthermore, less skilled subjects are much more likely to relate variables explicitly to information in the problem.

These models have proved to have considerable flexibility. First, we have been able to match errors and recoveries of one novice subject by simply deleting the output of a production corresponding to writing an equation miswritten or apparently forgotten by the subject. Second, the schema for representing problem information is sufficiently flexible to allow the representation of a very different kind of physics problem (from thermodynamics), and a more complex mechanics problem involving more than one context. Finally, we have been able to let the models solve these more difficult problems simply by adding small additional pieces of knowledge about how to relate different contexts.

These indications of easy extensibility give us hope that this kind of model, a very simple production system capturing gross features of human behavior, will be a fruitful vehicle for further work exploring how knowledge develops, i.e., how a novice solver becomes competent.

REFERENCES

Bhaskar, R. & Simon, H. A. Problem solving in semantically rich domains: An example from engineering thermodynamics. *Cognitive Science*, 1977, *1*, 193–215.

Bundy, A., Byrd, L., Luger, G., Mellish, C. & Palmer, M. Solving mechanics problems using meta-level inference. *Proceedings of the Sixth International Joint Conference on Artificial Intelligence*, 1979, 2,.

Bundy, A. *Analysing mathematical proofs, or reading between the lines*. Research Report 2, University of Edinburgh, Department of Artificial Intelligence, May 1975.

de Kleer, J. Qualitative and Quantitative Knowledge in Classical Mechanics. Master's thesis, MIT, December, 1975.

Forgy, C., & McDermott, J. *The OPS2 reference manual*. Technical Report, Computer-Science Department, Carnegie-Mellon University, 1978.

Forgy, C. L. *Implementing a fast syntactic pattern matcher*. Technical Report, Computer Science Department, Carnegie-Mellon University, August 1979.

Larkin, J. H. *Problem solving in physics*. Technical Report, Group in Science and Mathematics Education, University of California, Berkeley, 1977.

Larkin, J. H. Enriching formal knowledge: A model for learning to solve textbook physics problems. In Anderson, J. (Ed.), *Cognitive Skills and their Acquisition*. Hillsdale, N.J.: Lawrence Erlbaum, 1981.

Luger, G. F. *A comparison of human and automatic problem solving in two areas of mechanics*. Technical Report, Department of Artificial Intelligence, University of Edinburgh, 1979.

McDermott, J. & Forgy, C. Production system conflict resolution strategies. In Waterman, D. & Hayes-Roth, F. (Eds.), *Pattern-directed inference systems*, New York: Academic Press, 1978.

McDermott, J. & Larkin, J. H. *Rerepresenting textbook physics problems*, pages 156-164. The Canadian society for computational studies of intelligence, 1978. University of Toronto press.

Newell, A. Production systems: Models of control structures. In Chase, W. G. (Ed.), *Visual information processing*, New York: Academic Press, 1973.

Novak, G. S. Computer Understanding of Physics Problems Stated in Natural Language. PhD thesis, The University of Texas at Austin, March, 1976.

Novak, G. Representations of knowledge in a program for solving physics problems. *International Joint Conference on Artificial Intelligence*, 1977, *5*, 286–291.

Schneider, W. & Shiffrin, R. M. Controlled and automatic human information Processing: 1. detection, search, and attention. *Psychological Review*, 1977, *84*(1), 1–66.

Schiffrin, R. M. & Schneider, W. Controlled and automatic human information processing: II. perceptual learning, automatic attending, and a general theory. *Psychological Review*, 1977, *84*(2), 127–190.

Simon, D. P. & Simon, H. A. Individual differences in solving physics problems. In Siegler, R. (Ed.), *Children's thinking: What develops?, Hillsdale, N.J.: Lawrence Erlbaum Associates, 1978*.